

Dynamical Symmetry Breaking in Supersymmetric $SU(n_c)$ and $USp(2n_c)$ Gauge Theories

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ABSTRACT: We find the phase and flavor symmetry breaking pattern of each $N = 1$ supersymmetric vacuum of $SU(n_c)$ and $USp(2n_c)$ gauge theories, constructed from the exactly solvable $N = 2$ theories by perturbing them with small adjoint and generic bare hypermultiplet (quark) masses. In $SU(n_c)$ theories with $n_f \leq n_c$ the vacua are labelled by an integer r , in which the flavor $U(n_f)$ symmetry is dynamically broken to $U(r) \times U(n_f - r)$ in the limit of vanishing bare hyperquark masses. In the $r = 1$ vacua the dynamical symmetry breaking is caused by the condensation of magnetic monopoles in the $\underline{n_f}$ representation. For general r , however, the monopoles in the $\underline{n_f C_r}$ representation, whose condensation could explain the flavor symmetry breaking but would produce too-many Nambu–Goldstone multiplets, actually “break up” into “magnetic quarks”: the latter with nonabelian interactions condense and induce confinement and dynamical symmetry breaking. In $USp(2n_c)$ theories with $n_f \leq n_c + 1$, the flavor $SO(2n_f)$ symmetry is dynamically broken to $U(n_f)$, but with no description in terms of a weakly coupled local field theory. In both $SU(n_c)$ and $USp(2n_c)$ theories, with larger numbers of quark flavors, besides the vacua with these properties, there exist also vacua in free magnetic phase, with unbroken global symmetry.

1 Introduction and Summary

Many beautiful exact results on supersymmetric $4D$ gauge theories have been obtained recently, following Seiberg and Witten's breakthrough on $N = 2$ supersymmetric theories [1]-[3]. One exciting related development is Seiberg's $N = 1$ non-Abelian duality, present in many cases, between a pair of theories with different number of color, flavor, and matter contents, which describe the same low-energy physics [4]-[6]. Another is the discovery of universal classes of conformally invariant theories (CFT) for different values of color, flavor, and in some cases, for appropriately tuned values of the parameters of the theory [4]-[9].

Still another concerns the microscopic mechanism of confinement (e.g., monopole condensation) and dynamical symmetry breaking, and study of other phases such as the oblique confinement, upon addition of a perturbation breaking the $N = 2$ supersymmetry to $N = 1$ and/or to $N = 0$ [1, 2],[10]-[13]. In fact, an interesting phenomenon has been observed in the case of $SU(2)$ gauge theories with various flavors and with adjoint mass perturbation [1, 2]: confinement is caused by condensation of magnetic monopoles carrying nontrivial flavor quantum numbers. More explicitly, for $n_f = 2$, monopoles in the $(\underline{2}, \underline{1}) + (\underline{1}, \underline{2})$ (spinor) representation of the flavor $[SU(2) \times SU(2)]/Z_2 = SO(4)$ group is found to condense upon $N = 1$ perturbation $\mu \text{Tr } \Phi^2$: the flavor symmetry is necessarily broken to $U(2)$. For $n_f = 3$, monopoles in the $\underline{4}$ (spinor) representation of the flavor $SO(6)$ group condense with $\mu \neq 0$ and the flavor symmetry is broken to $U(3)$. In such systems spontaneous global symmetry breaking is caused by the same dynamical mechanism responsible for confinement.

In one of the vacua with $n_f = 3$, though, the condensing entity carries magnetic number twice the basic unit but is flavor neutral. In this case - which can be interpreted as oblique confinement à la 't Hooft [14] - confinement is not accompanied by flavor symmetry breaking. For $n_f = 1, 4$, there is no dynamical flavor symmetry breaking.

These results naturally lead to a conjecture that the condensation of magnetic monopoles with non-trivial flavor transformation property might in a general class of systems explain the confinement à la 't Hooft, Nambu, Mandelstam [14, 15] and the flavor symmetry breaking, simultaneously¹. However, a simple thought reveals a problem with this picture. For instance, the monopoles in $USp(2n_c)$ theories transform under the spinor representation of $SO(2n_f)$ flavor symmetry, and their effective low-energy Lagrangian coupled to the magnetic $U(1)$ gauge group would have a large accidental $SU(2^{n_f-1})$ flavor symmetry: their condensation would lead to far too many Nambu-Goldstone multiplets. The case of $SU(2)$ gauge theories was special because the flavor symmetries of the monopole action precisely coincide with the symmetry of the microscopic theories, somewhat accidentally, due to the small number of flavors. It is not at all obvious how such a paradox is avoided in higher-rank theories.

Argyres, Plesser and Seiberg [17] studied higher-rank $SU(n_c)$ theories with $n_f \leq 2n_c - 1$ (asymptotically free) in detail. They showed how the non-renormalization theorem of the hyperKähler metric on the Higgs branch could be used to show the persistence of unbroken non-abelian gauge group at the "roots" of the Higgs branches (non-baryonic and baryonic branches) where they intersect the

¹Such a possibility has been critically discussed recently in QCD [16].

Coulomb branch. Some isolated points on the non-baryonic roots with $SU(r)$ ($r \leq [n_f/2]$) gauge group as well as the baryonic root (single point) with $SU(\tilde{n}_c) = SU(n_f - n_c)$ gauge group were found to survive the $\mu \neq 0$ perturbation. Their main focus, however, was an attempt to “derive” Seiberg’s duality between $SU(n_c)$ and $SU(\tilde{n}_c)$ gauge theories relying on the so-called baryonic root. Their “derivation,” however, was incomplete as it did not produce all components of the “meson” superfield. The effective low-energy theory was perturbed by a relevant operator (the mass term for the mesons) and did not flow to the Seiberg’s $N = 1$ magnetic theory correctly.² On the other hand, the issue of flavor symmetry breaking was not studied at any depth in [17]. Their analysis also left a puzzle why there were “extra” theories at the non-baryonic roots which seemingly had nothing to do with Seiberg’s dual theories. Another paper by Argyres, Plesser and Shapere addressed similar questions, and left puzzles, in $SO(n_c)$ and $USp(2n_c)$ theories [18].

We investigate here the microscopic mechanism of dynamical symmetry breaking, taking as theoretical laboratory the same class of theories studied by the above mentioned authors [17, 18], namely, theories constructed from exactly solvable $N = 2$ $SU(n_c)$ and $USp(2n_c)$ gauge theories with all possible numbers of flavor compatible with asymptotic freedom, by perturbing them with a small adjoint mass (reducing supersymmetry to $N = 1$) and quark masses. The Lagrangian of the models has the structure

$$\mathcal{L} = \frac{1}{8\pi} \text{Im } \tau_{cl} \left[\int d^4\theta \Phi^\dagger e^V \Phi + \int d^2\theta \frac{1}{2} WW \right] + \mathcal{L}^{(quarks)} + \Delta\mathcal{L} + \Delta'\mathcal{L}, \quad (1.1)$$

where

$$\Delta\mathcal{L} = \int d^2\theta \mu \text{Tr } \Phi^2 \quad (1.2)$$

reduces the supersymmetry to $N = 1$;

$$\mathcal{L}^{(quarks)} = \sum_i \left[\int d^4\theta \{Q_i^\dagger e^V Q_i + \tilde{Q}_i^\dagger e^{\tilde{V}} \tilde{Q}_i\} + \int d^2\theta \{\sqrt{2}\tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i Q_i\} \right] \quad (1.3)$$

describes the n_f flavors of hypermultiplets (“quarks”), and

$$\tau_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2} \quad (1.4)$$

is the bare θ parameter and coupling constant. The $N = 1$ chiral and gauge superfields $\Phi = \phi + \sqrt{2}\theta\psi + \dots$, and $W_\alpha = -i\lambda + \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu)_\alpha^\beta F_{\mu\nu} \theta_\beta + \dots$ are both in the adjoint representation of the gauge group, while the hypermultiplets are taken in the fundamental representation of the gauge group. We shall consider, besides the adjoint mass, small *generic* nonvanishing bare masses for the hypermultiplets (“quarks”). The advantage of doing so is that the only vacua retained are those in which the gauge coupling constant grows in the infrared. Another advantage is that all flat directions are eliminated in this way and one is left with a finite number of isolated vacua; keeping track of this number allows us to do highly nontrivial checks of our analyses at various stages.

The most salient features of the result of our analysis will be as follows. The theories studied in different regimes, semiclassical, large and small adjoint and/or bare quark masses, give a mutually

² We thank P. Argyres and A. Buchel for discussions on this point.

consistent picture as regards the number of the vacua and the dynamical properties in each of them. Various dynamical possibilities, found to be realized in the $SU(n_c)$ and $USp(2n_c)$ theories with a small finite adjoint mass and in the vanishing bare quark mass limit, are summarized in Tables 1 and 2.

label (r)	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
0 (NB)	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f)$
1 (NB)	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f - 1) \times U(1)$
$2, \dots, [\frac{n_f-1}{2}]$ (NB)	dual quarks	$SU(r) \times U(1)^{n_c-r}$	Confinement	$U(n_f - r) \times U(r)$
$n_f/2$ (NB)	rel. nonloc.	-	Almost SCFT	$U(n_f/2) \times U(n_f/2)$
BR	dual quarks	$SU(\tilde{n}_c) \times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$U(n_f)$

Table 1: Phases of $SU(n_c)$ gauge theory with n_f flavors. “rel. nonloc.” means that relatively nonlocal monopoles and dyons coexist as low-energy effective degrees of freedom. “Confinement” and “Free Magnetic” refer to phases with $\mu \neq 0$. “Almost SCFT” means that the theory is a non-trivial superconformal one when $\mu = 0$ but confines with $\mu \neq 0$. NB and BR stand for “nonbaryonic roots” and “baryonic roots” (see Sec. 8) respectively. $\tilde{n}_c \equiv n_f - n_c$.

With small but generic bare quark masses, the order parameter of confining vacua is indeed the condensation of magnetic monopoles for every $U(1)$ factor à la ’t Hooft, in all cases considered. The massless limit, however, is non-trivial and exhibits a much richer range of interesting dynamical possibilities.

In $SU(n_c)$ theories with n_f flavors, the following diversity of dynamical scenarios are realized, according to the number of flavors and to the particular vacua considered. In the first group of vacua with finite meson or dual quark vacuum expectation values (VEVS), labelled by an integer r , $r \leq [n_f/2]$, the system is in confinement phase. The nature of the actual carrier of the flavor quantum numbers depends on r . In vacua with $r = 0$, magnetic monopoles are singlets of the global $U(n_f)$ group, hence no global symmetry breaking accompanies confinement. This is analogous to the oblique confinement ’t Hooft suggested for QCD around $\theta = \pi$.

In vacua with $r = 1$, the light particles are magnetic monopoles in the fundamental representation of $U(n_f)$ flavor group, and are charged under one of the color $U(1)$ factors. Their condensation leads to the confinement and flavor symmetry breaking, simultaneously.

In vacua labelled by r , $2 \leq r < n_f/2$ but $r \neq n_f - n_c$, the grouping of the associated singularities on the Coulomb branch, with multiplicity, ${}_n C_r$,³ at first sight suggests the condensation of monopoles in the rank- r anti-symmetric tensor representation of the global $SU(n_f)$ group. Actually, this does not occur. The true low-energy degrees of freedom of these theories are (magnetic) quarks plus a number of singlet monopoles of an effective $SU(r) \times U(1)^{n_c-r}$ gauge theory. Monopoles in

³In this paper, we use the traditional notation ${}_n C_r$ for the binomial coefficient most of time: another frequently used symbol is $\binom{n}{r}$.

higher representations of $SU(n_f)$ flavor group probably exist semi-classically as seen in a Jackiw–Rebbi type analysis [19]. Such monopoles can be interpreted as “baryons” made of the magnetic quarks, which, interactions being infrared-free, however break up into magnetic quarks before they become massless at singularities on the Coulomb branch. It is the condensation of these magnetic quarks that induces confinement and flavor symmetry breaking, $U(n_f) \rightarrow U(r) \times U(n_f - r)$, in these vacua. The system thus realizes the exact global symmetry of the theory in a Nambu-Goldstone mode, without having unusually many Nambu-Goldstone bosons. It is a novel mechanism for confinement and dynamical symmetry breaking.

In the special cases with $r = n_f/2$, still another dynamical scenario takes place. In these cases, the interactions among the monopoles become so strong that the low-energy theory describing them is a nontrivial superconformal theory, with conformal invariance explicitly broken by the adjoint or quark masses. Although the symmetry breaking pattern is known ($U(n_f) \rightarrow U(n_f/2) \times U(n_f/2)$), the low-energy degrees of freedom involve relatively nonlocal fields and their interactions cannot be described in terms of a local action.

Finally, in the group of vacua labelled by $r = n_f - n_c$, the low-energy degrees of freedom are again magnetic quarks and a number of singlet monopoles of an effective infrared-free $SU(n_f - n_c) \times U(1)^{2n_c - n_f}$ gauge theory. There are two physically distinct sub-groups of vacua: one in which the magnetic quarks condense (i.e. confinement phase) with the unbroken symmetry $U(n_f - n_c) \times U(n_c)$ is analogous to the ones found for generic r ’s; the other is vacua in which magnetic-quark do not condense and remain as physically observable particles at long distances (free magnetic phase). The global $U(n_f)$ symmetry remains unbroken.

This last phase is related to the one discovered by Seiberg in $N = 1$ massless SQCD for the range of n_f , $n_c + 1 < n_f < 3n_c/2$. Nevertheless, it should be emphasized that the $m_i \rightarrow 0$ limit here is a smooth one and the symmetry properties of the vacua are independent of the way the limit is taken, while in SQCD without the adjoint chiral superfield the vacuum properties depend critically on the order in which the m_i ’s approach zero, showing typically the phenomenon of the run-away vacua.

All in all, we find the number

$$\mathcal{N}_1 = (2n_c - n_f) 2^{n_f - 1} \quad (1.5)$$

of $N = 1$ vacua with finite flavor-carrying VEVs and

$$\mathcal{N}_2 = \sum_{r=0}^{n_f - n_c - 1} (n_f - n_c - r) \cdot n_f C_r, \quad (1.6)$$

of them with vanishing VEVs. The latter is present only for theories with the large number of flavors ($n_f \geq n_c + 1$). Their sum, $\mathcal{N} = \sum_{r=0}^{\min\{n_f, n_c - 1\}} n_f C_r (n_c - r)$, correctly generalizes⁴ the well-known number of the vacua in the $SU(2)$ gauge theory, $\mathcal{N}_{SU(2)} = n_f + 2$.

In $USp(2n_c)$ theories, again, we find two groups of vacua, whose properties are shown in Table 2. The most interesting difference as compared to the $SU(n_c)$ theory is that here the entire first

⁴ It may be said that the fact that the generalization is given by these formulas and analogous ones Eq.(1.7), Eq.(1.8), and not simply, e.g., by $n_f + n_c$, reveals the richness of dynamical scenarios of these theories.

	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
1st Group	rel. nonloc.	-	Almost SCFT	$U(n_f)$
2nd Group	dual quarks	$USp(2\tilde{n}_c) \times U(1)^{n_c - \tilde{n}_c}$	Free Magnetic	$SO(2n_f)$

Table 2: Phases of $USp(2n_c)$ gauge theory with n_f flavors with $m_i \rightarrow 0$. $\tilde{n}_c \equiv n_f - n_c - 2$.

Label (r)	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
0	monopoles	$U(1)^{n_c}$	Confinement	$U(n_f)$
1	monopoles	$U(1)^{n_c}$	Confinement	$U(n_f - 1) \times U(1)$
$2, \dots, [\frac{n_f - 1}{2}]$	dual quarks	$SU(r) \times U(1)^{n_c - r + 1}$	Confinement	$U(n_f - r) \times U(r)$
$n_f/2$	dual quarks	$SU(n_f/2) \times U(1)^{n_c - n_f/2 + 1}$	Almost SCFT	$U(n_f/2) \times U(n_f/2)$

Table 3: The first group of vacua of $USp(2n_c)$ theory with n_f flavors with $m_i = m \neq 0$.

group of vacua correspond to SCFT. As the superconformal theory is a nontrivial one, one does not have a local effective Lagrangian description for those theories. Nonetheless, the symmetry breaking pattern can be deduced, from the analysis done at large μ : $SO(2n_f)$ symmetry is always spontaneously to $U(n_f)$.

It is most instructive to consider the equal but nonvanishing quark mass case, first. (See Table 3.) The flavor symmetry group of the underlying theory is now broken explicitly to $U(n_f)$. The first group of vacua split into various branches labelled by r , $r = 0, 1, 2, \dots, [\frac{n_f - 1}{2}]$, each of which is described by a *local* effective gauge theory of Argyres-Plesser-Seiberg [17], with gauge group $SU(r) \times U(1)^{n_c - r + 1}$ and n_f (dual) quarks in the fundamental representation of $SU(r)$. Indeed, the gauge invariant composite VEVs characterizing these theories differ by some powers of m , and the validity of each effective theory is limited by small fluctuations of order of m around each vacuum. In the limit $m \rightarrow 0$ these points in the quantum moduli space (QMS) collapse into one single point. Obviously, a smooth $m_i \rightarrow 0$ limit is not possible. The location of this singularity can be obtained exactly in terms of Chebyshev polynomials. At the singularity there are mutually non-local dyons and hence the theory is at a non-trivial infrared fixed point. In the example of $USp(4)$ theory with $n_f = 4$, we have explicitly verified this by determining the singularities and branch points at finite equal mass m and then by studying the limit $m \rightarrow 0$.

These cases, together with the special $r = n_f/2$ nonbaryonic root for the $SU(n_c)$ theory, constitute another new mechanism for dynamical symmetry breaking: although the global symmetry breaking pattern deduced indirectly looks familiar enough, the low-energy degrees of freedom are relatively nonlocal dual quarks and dyons. It would be interesting to get a better understanding of this phenomenon.

For large numbers of flavor, there are also vacua, just as in large n_f $SU(n_c)$ theories, with no confinement and no dynamical flavor symmetry breaking. The low-energy particles are solitonlike

magnetic quarks which weakly interact with dual (in general) non-abelian gauge fields: the system is in the free magnetic phase.

The vacuum counting gives, for $USp(2n_c)$ theories,

$$\mathcal{N}_1 = (2n_c + 2 - n_f) \cdot 2^{n_f-1} \quad (1.7)$$

vacua with finite VEVs (first group of vacua), and

$$\mathcal{N}_2 = \sum_{r=0}^{n_f-n_c-2} (n_f - n_c - 1 - r) \cdot n_f C_r \quad (1.8)$$

of them with vanishing VEVs (second group of vacua); the latter are present only for $n_f \geq n_c + 2$.

The paper will be organized as follows. After discussing briefly the standard expectation on chiral symmetry breaking in Sec. 2, we start in Sec. 3 a preparatory analysis, finding all isolated semi-classical vacua by minimizing the scalar potential and determining the vevs of the adjoint scalar and of the squark fields in each of them. This allows us to count the number of all possible vacua of the theory, after taking appropriate account of Witten's index corresponding to the unbroken gauge group in each case.

The first major step of our analysis is the analytic, first-principle determination of the global symmetry breaking pattern in each vacuum, done by studying the theories at *large* μ ($\gg \Lambda$), and $m_i \rightarrow 0$ (Sec. 4). Such a determination is possible because in this case the effective superpotential can be read off from the bare Lagrangian by integrating out the heavy, adjoint fields and by adding to it the known exact instanton-induced superpotentials of the corresponding $N = 1$ theories. By minimizing the scalar potential, we reproduce in all cases the correct number of the vacua and explicitly determine the pattern of global symmetry breaking in each them. By $N = 1$ supersymmetry and holomorphic dependence of physics on μ the same symmetry breaking pattern is valid at any finite μ .

In Section 5 another related limit, $\mu \rightarrow \infty$, m_i and $\Lambda_1 \equiv \mu^{\frac{n_c}{3n_c-n_f}} \Lambda^{\frac{2n_c-n_f}{3n_c-n_f}}$ fixed, is studied and consistency with the known results in the standard $N = 1$ theories without adjoint fields is checked.

The properties of the quantum vacua at *small* μ and *small* m_i are studied in detail, in Sections 6 - 9. In Section 6 we show how all of the $N = 1$ vacua arise from various classes of conformally invariant theories (CFT) upon perturbation in bare quark masses on Seiberg-Witten curves, reproducing the correct number of $N = 1$ vacua found in the earlier analyses. In Section 7 we check and illustrate these results in the cases of rank-two gauge groups, $SU(3)$ and $USp(4)$ theories, by directly finding the associated singularities numerically. In Section 8 the effective-Lagrangian description of these $N = 1$ theories is analysed, where we verify that the number of $N = 1$ vacua and the symmetry breaking pattern in each of them indeed agree with what we found earlier in Sections 3-7. This leads us to a better, more microscopic understanding of the phenomena of confinement and dynamical global symmetry breaking, as summarized above. The actual perturbation theory in bare quark masses on Seiberg-Witten curves, whose outcome is quite central to the whole analysis but whose analysis *per se* is independent of the rest of the paper, is developed in the last section (Sec. 9).

Several technical discussions are relegated to Appendices. Appendix A gives a proof of $SO(2N) \cap USp(2N) = U(N)$; in Appendix B we discuss the Jackiw-Rebbi construction of flavor multiplet structure of semiclassical monopoles for $SU(n_c)$ and $USp(2n_c)$ gauge theories; we list explicit expressions for a_{Di} , a_i , $\frac{\partial a_{Di}}{\partial u_j}$, and $\frac{\partial a_i}{\partial u_j}$ in Appendix C; the proof of absence of the “nonbaryonic branch root” with $r = \tilde{n}_c = n_f - n_c$ is given in Appendix D; the study of the monodromy around the seventeen singularities of $SU(3)$, $n_f = 4$ theory is discussed in Appendix E .

A shorter version of this work has appeared already [20]. The case of $SO(n_c)$ theories, where some new subtleties are present, will be discussed in a separate article.

2 Expected Pattern of Chiral Symmetry Breaking

In non-supersymmetric theories with fermions in the fundamental representation of the gauge group, global symmetry can be broken spontaneously if a fermion bilinear condensate

$$\langle \psi\psi \rangle \sim \Lambda^3 \quad (2.1)$$

forms. Apart from the requirement of gauge invariance there are no general rules which fermion pairs condense, although in QCD at large n_c limit one can argue [21] that $SU_L(n_f) \times SU_R(n_f) \times U_V(1)$ symmetry is broken to the diagonal $SU_V(n_f) \times U_V(1)$, which is the observed pattern of chiral symmetry breaking in Nature.

Nonetheless certain general considerations can be made. For nonsupersymmetric $SU(2)$ theories, $2n_f$ fermions transform as the fundamental representation of the global $SU(2n_f)$ group: bilinear condensate ($a, b = 1, 2$ are color indices; $i, j = 1, 2, \dots, 2n_f$ are flavor indices), $\langle \epsilon^{ab} \psi_a^i \psi_b^j \rangle$, is necessarily antisymmetric in (i, j) . If these condensates can be put by an $SU(2n_f)$ transformation into the “standard” form

$$\langle \epsilon^{ab} \psi_a^i \psi_b^j \rangle = \text{const. } J^{ij}, \quad (2.2)$$

where $J = i\sigma_2 \otimes \mathbf{I}_{n_f \times n_f}$, then such condensates would leave $USp(2n_f)$ invariant. Note that this is consistent with what was found in the $N = 2$ $SU(2)$ QCD, broken to $N = 1$ by a small adjoint mass term, in case of $n_f = 2$ and in one of the vacua of $n_f = 3$, where the symmetry of the vacuum was found to be $U(n_f)$ [2, 13]. The reason is that in the model of [2, 13] the global symmetry of the theory is reduced to $SO(2n_f)$ due to the Yukawa interaction, and the intersection of $USp(2n_f)$ and $SO(2n_f)$ is precisely $U(n_f)$ (see Appendix A). For another vacuum of the $n_f = 3$ theory (where oblique confinement of ’t Hooft takes place), the monopole that condenses is a flavor singlet and chiral symmetry remains unbroken.

In the standard QCD with $SU(n_c)$ gauge symmetry ($n_c \geq 3$), $2n_f$ fermions transform as $(\underline{n}_f, \underline{1}) + (\underline{1}, \underline{n}_f^*)$ of $SU_L(n_f) \times SU_R(n_f)$; condensates of the form

$$\langle \bar{\psi}_R^i \psi_{Lj} \rangle = v \delta_j^i, \quad (2.3)$$

is believed to form, at least for small n_f , leaving the unbroken diagonal $SU(n_f)$ symmetry. Unfortunately, in the corresponding $N = 2$ theories (with a small $N = 1$ perturbation) the axial symmetry is explicitly broken at the tree level by the characteristic Yukawa interactions so that the global symmetry contains only the diagonal $SU(n_f)$, already at the tree level. Thus $SU(n_c)$ theories will be considered below mainly as a testing ground of our approach, in correctly identifying the quantum vacua which survive $N = 1$ perturbation, matching the numbers of classical and quantum vacua, and in verifying in each such vacua the ’t Hooft–Mandelstam mechanism for confinement. In fact this study reveals new, unexpected ways confinement and dynamical symmetry breaking are realized in non-Abelian gauge theories.

The cases of $USp(2n_c)$ theories are more promising. As noted above for $SU(2)$ gauge theories, in a nonsupersymmetric theories with $2n_f$ fermions in the fundamental representation, the global

symmetry is $SU(2n_f)$. Bifermion condensates of the “standard” form

$$\langle \psi_i^a \psi_j^b J_{ab} \rangle = v J_{ij} \quad (2.4)$$

would break it to $USp(2n_f)$.

On the other hand, the corresponding $N = 2$ models (1.1) have a smaller flavor group, $SO(2n_f) \subset SU(2n_f)$ due to the Yukawa interactions. Nevertheless, there is a nontrivial overlap between its flavor group $SO(2n_f)$ and the $USp(2n_f)$ (expected invariance group for nonsupersymmetric model), which is $U(n_f)$. One then expects that the global chiral symmetry $SO(2n_f)$ is broken spontaneously to $U(n_f) \subset SO(2n_f)$. We shall see below that these expectations are indeed met by quantum vacua of the $USp(2n_c)$ theories (in the large flavor cases, these take place in the first group of vacua, while we find also a second group of vacua in which the chiral $SO(2n_f)$ symmetry remains unbroken). The proof that $SO(2N) \cap USp(2N) = U(N)$ is given in Appendix A.

3 Semi-Classical Vacua

In this section, we find all semi-classical vacua in $N = 2$ $SU(n_c)$ and $USp(2n_c)$ theories with quark hypermultiplets in their fundamental representations, perturbed by the quark masses as well as that of the adjoint fields in the $N = 2$ vector multiplet. The analysis is quantum mechanically valid at large μ and m_i . $N = 1$ supersymmetry and holomorphy in μ and m_i forbid any phase transition as one moves to smaller $|m_i|$ and $|\mu|$ hence the same number of $N = 1$ vacua must be present in the different regimes to be considered in the subsequent sections. In particular, this analysis allows to determine the symmetry breaking pattern in the equal (and nonvanishing) mass case, in which the classical global symmetry is $U(n_f)$ in both $SU(n_c)$ and $USp(2n_c)$ theories.

3.1 Semi-Classical Vacua in $SU(n_c)$

In the limit $m_i \rightarrow 0$ and $\mu \rightarrow 0$, the global symmetry of the model is $U(n_f) \times Z_{2n_c-n_f} \times SU_R(2)$. The superpotential of our theory, with $N = 2$ supersymmetry softly broken to $N = 1$ by the adjoint mass, is given by:

$$W = \mu \text{Tr} \Phi^2 + \sqrt{2} \tilde{Q}_i^a \Phi_a^b Q_b^i + m_i \tilde{Q}_i^a Q_a^i, \quad (3.1)$$

where

$$\Phi \equiv \lambda^A \Phi^A, \quad (A = 1, 2, \dots, N^2 - 1), \quad (3.2)$$

$$\text{Tr}(\lambda^A \lambda^B) = \frac{1}{2} \delta^{AB}. \quad (3.3)$$

$i = 1, 2, \dots, n_f$ is the flavor index; $a, b = 1, 2, \dots, n_c$ are the color indices. Note that with our normalization of the $SU(n_c)$ generators the color Fierz relation reads

$$\sum_{A=1}^{n_c^2-1} (\lambda^A)_c^d (\lambda^A)_a^b = \frac{1}{2} \left[\delta_c^b \delta_a^d - \frac{1}{n_c} \delta_c^d \delta_a^b \right]. \quad (3.4)$$

The vacuum equations are

$$[\Phi, \Phi^\dagger] = 0; \quad (3.5)$$

$$\nu \delta_a^b = Q_a^i (Q^\dagger)_i^b - (\tilde{Q}^\dagger)_a^i \tilde{Q}_i^b; \quad (3.6)$$

$$Q_a^i \tilde{Q}_i^b - \frac{1}{n_c} \delta_a^b (Q_c^i \tilde{Q}_i^c) + \sqrt{2} \mu \Phi_a^b = 0; \quad (3.7)$$

$$Q_a^i m_i + \sqrt{2} \Phi_a^b Q_b^i = 0 \quad (\text{no sum over } i); \quad (3.8)$$

$$m_i \tilde{Q}_i^a + \sqrt{2} \tilde{Q}_i^b \Phi_a^b = 0 \quad (\text{no sum over } i). \quad (3.9)$$

where the quark masses have been taken diagonal by flavor rotations.

Use first $SU(n_c)$ rotation to bring Φ into diagonal form,

$$\Phi = \text{diag}(\phi_1, \phi_2, \dots, \phi_{n_c}), \quad \sum \phi_a = 0. \quad (3.10)$$

Eq. (3.8) and Eq. (3.9) say that Q_a^i and \tilde{Q}_i^b are either nontrivial eigenvectors of the matrix Φ with possible eigenvalues m_i , or null vectors. With Φ put in the diagonal form, and with generic (so unequal) masses, the eigenvectors have simple forms,

$$Q^i = (0, \dots, d_i, 0, \dots), \quad (3.11)$$

each with only one nonvanishing component (similarly for \tilde{Q}_i^a). There can be at most n_f nontrivial eigenvalues, chosen from m_1, m_2, \dots, m_{n_f} ; at the same time the form (3.10) allows for at most $n_c - 1$ nonzero independent elements of Φ . The solutions can thus be classified by the number of nontrivial eigenvectors, $r = 1, 2, \dots, \min\{n_f, n_c - 1\}$. There are $\binom{n_f}{r}$ solutions for a given r , according to which m_i 's appear as eigenvalues.

The solution with eigenvalues m_1, m_2, \dots, m_r is ⁵:

$$Q_a^i = \begin{pmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ d_r \\ 0 \\ \vdots \end{pmatrix}; \quad Q_a^i = 0, \quad i = r + 1, \dots, n_f. \quad (3.12)$$

$$\tilde{Q}_i^a = \begin{pmatrix} \tilde{d}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{d}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ \tilde{d}_r \\ 0 \\ \vdots \end{pmatrix}; \quad \tilde{Q}_i^a = 0, \quad i = r + 1, \dots, n_f, \quad (3.13)$$

where

$$r = 0, 1, \dots, \min\{n_f, n_c - 1\}, \quad (3.14)$$

$$\Phi = \frac{1}{\sqrt{2}} \text{diag}(-m_1, -m_2, \dots, -m_r, c, \dots, c); \quad c = \frac{1}{n_c - r} \sum_{k=1}^r m_k. \quad (3.15)$$

d_i 's can be chosen real (by residual $SU(n_c)$); \tilde{d}_i 's are in general complex. They are given by Eqs. (3.21) and (3.18) below.

Eq. (3.5) is obviously satisfied. The nondiagonal ($a \neq b$) of Eq. (3.6) is also obvious. The first r diagonal ($a = b$) equations are:

$$\nu = d_i^2 - |\tilde{d}_i|^2; \quad (i = 1, 2, \dots, r); \quad (3.16)$$

the others give

$$\nu = 0, \quad (3.17)$$

hence

$$d_i^2 = |\tilde{d}_i|^2. \quad (3.18)$$

⁵ These results are slight generalization of the ones in [2, 3, 17, 22] to generic nonvanishing quark and adjoint masses. Note that the flat directions are completely eliminated.

Eq. (3.8) and Eq. (3.9) are satisfied by construction; Eq. (3.7) gives for $a = b = 1, 2, \dots, r$:

$$d_i \tilde{d}_i - \frac{1}{n_c} \sum_k d_k \tilde{d}_k = \mu m_i; \quad (3.19)$$

from which one finds that

$$\sum_k d_k \tilde{d}_k = \frac{n_c}{n_c - r} \mu \sum_{k=1}^r m_k \quad (3.20)$$

and

$$d_i \tilde{d}_i = \mu m_i + \frac{1}{n_c - r} \mu \sum_{k=1}^r m_k \quad (d_i > 0). \quad (3.21)$$

On the other hand, Eq. (3.7) for $a = b = r + 1, r + 2, \dots, n_c$ gives

$$\sum_k d_k \tilde{d}_k = n_c \mu c. \quad (3.22)$$

This is compatible with Eq. (3.20) because of (3.15).

A solution with a given r leaves a local $SU(n_c - r)$ invariance. Thus each of them counts as a set of $n_c - r$ solutions (Witten's index). In all, therefore, there are

$$\mathcal{N} = \sum_{r=0}^{\min\{n_f, n_c-1\}} (n_c - r) \binom{n_f}{r} \quad (3.23)$$

classical solutions. (For $r = 0$, $Q_a^i = \tilde{Q}_i^b = 0$, $\Phi = 0$ is obviously a solution with full $SU(n_c)$ invariance.)

For $n_c = 2$ the formula (3.23) reproduces the known result ($\mathcal{N} = 2 + n_f$) as can be easily verified.

Note that when n_f is equal to or less than n_c the sum over r is done readily, and Eq. (3.23) is equivalent to

$$\mathcal{N}_1 = (2n_c - n_f) 2^{n_f-1}, \quad (n_f \leq n_c). \quad (3.24)$$

3.2 Semi-Classical Vacua in $USp(2n_c)$

The superpotential reads in this case

$$W = \mu \text{Tr} \Phi^2 + \frac{1}{\sqrt{2}} Q_a^i \Phi_b^a Q_c^i J^{bc} + \frac{m_{ij}}{2} Q_a^i Q_b^j J^{ab}, \quad (3.25)$$

where $J = i\sigma_2 \otimes \mathbf{1}_{n_c}$ and

$$m = -i\sigma_2 \otimes \text{diag}(m_1, m_2, \dots, m_{n_f}). \quad (3.26)$$

In the $m_i \rightarrow 0$ and $\mu \rightarrow 0$ limit, the global symmetry is $SO(2n_f) \times Z_{2n_c+2-n_f} \times SU_R(2)$.

The vacuum equations are:

$$[\Phi, \Phi^\dagger] = 0; \quad (3.27)$$

$$\sum_i (Q_a^{i\dagger} Q_b^i - Q_{n_c+b}^{i\dagger} Q_{n_c+a}^i) = 0; \quad \sum_i Q_a^{i\dagger} Q_{n_c+b}^i = 0; \quad (3.28)$$

$$\sqrt{2} \Phi_b^a Q_c^i J^{bc} + m_{ij} Q_b^j J^{ab} = 0. \quad (3.29)$$

$$2\mu \Phi_a^b + \frac{1}{\sqrt{2}} Q_a^i J^{bc} Q_c^i = 0. \quad (3.30)$$

To find the solutions of these equations diagonalize first Φ by a unitary transformation:

$$\Phi = \text{diag}(\phi_1, \phi_2, \dots, \phi_{n_c}, -\phi_1, -\phi_2, \dots, -\phi_{n_c}). \quad (3.31)$$

Define

$$\tilde{Q}_a^i \equiv Q_{n_c+a}^i. \quad (3.32)$$

The vacuum equations can be rewritten as:

$$\sum_i (Q_a^{i\dagger} Q_b^i - \tilde{Q}_b^{i\dagger} \tilde{Q}_a^i) = 0; \quad \sum_i Q_a^{i\dagger} \tilde{Q}_b^i = 0; \quad (i = 1, \dots, 2n_f); \quad (3.33)$$

and

$$\sqrt{2} \phi_a \tilde{Q}_a^i - m_i \tilde{Q}_a^{n_f+i} = 0; \quad (3.34)$$

$$\sqrt{2} \phi_a \tilde{Q}_a^{n_f+i} + m_i \tilde{Q}_a^i = 0; \quad (3.35)$$

$$\sqrt{2} \phi_a Q_a^i + m_i Q_a^{n_f+i} = 0; \quad (3.36)$$

$$\sqrt{2} \phi_a Q_a^{n_f+i} - m_i Q_a^i = 0; \quad (3.37)$$

$$Q_a^i \tilde{Q}_b^i = 0, \quad (a \neq b); \quad (3.38)$$

and

$$2\sqrt{2} \mu \phi_a + Q_a^i \tilde{Q}_a^i + Q_a^{n_f+i} \tilde{Q}_a^{n_f+i} = 0. \quad (3.39)$$

In Eq. (3.34) – Eq. (3.39) the index i runs only over $i = 1, 2, \dots, n_f$.

The solutions can again be classified by the number of the nonzero ϕ 's, $r = 1, 2, \dots, \min\{n_f, n_c\}$. The one with r masses m_1, \dots, m_r is

$$\phi = \frac{1}{\sqrt{2}} \text{diag}(im_1, im_2, \dots, im_r, 0, \dots, -im_1, -im_2, \dots, -im_r, 0, \dots); \quad (3.40)$$

$$Q_a^i = \begin{pmatrix} d_1 & & & & -id_1 & & & & \\ & d_2 & & & & -id_2 & & & \\ & & \ddots & & & & \ddots & & \\ & & & d_r & & & & -id_r & \\ & & & & 0 & & & & 0 \\ & & & & & \ddots & & & \\ -id_1 & & & & & & d_1 & & \\ & -id_2 & & & & & & d_2 & \\ & & \ddots & & & & & & \ddots \\ & & & -id_r & & & & & d_r \\ & & & & 0 & & & & 0 \\ & & & & & \ddots & & & \\ & & & & & & & & \ddots \end{pmatrix} \quad (3.41)$$

where

$$d_i = \sqrt{m_i \mu}. \quad (3.42)$$

Each solution with r leaves unbroken $USp(2(n_c - r))$ hence counts as $n_c - r + 2$ solutions. The total number of the classical vacua is then

$$\mathcal{N} = \sum_{r=0}^{\min\{n_c, n_f\}} (n_c - r + 1) \cdot \binom{n_f}{r}. \quad (3.43)$$

Note that for smaller values of n_f , the sum over r is easily done and an equivalent formula is

$$\mathcal{N} = (2n_c + 2 - n_f) 2^{n_f - 1}, \quad (n_f \leq n_c). \quad (3.44)$$

It is amusing (and reassuring) that different formulas, Eq.(3.23) , Eq.(3.24), Eq.(3.43) and Eq.(3.44) found here reproduce correctly the formula

$$\mathcal{N}_{SU(2)} = n_f + 2, \quad (3.45)$$

for the $SU(2)$ gauge theory (which is the simplest case, both of $SU(n_c)$ and $USp(2n_c)$), for $n_f = 0 \sim 4$.

4 Determination of Symmetry Breaking Patterns at Large μ

In this section, we determine the number of $N = 1$ vacua and the pattern of flavor symmetry breaking in each of them, in the limit $m_i \rightarrow 0$. This can be done most easily by studying these $SU(n_c)$ and $USp(2n_c)$ theories at large adjoint mass μ . The advantage of considering the theories at large μ is that the adjoint field can be integrated out from the theory: the resulting low-energy effective theory is an exactly known $N = 1$ supersymmetric gauge theory, perturbed by certain superpotential terms suppressed by $1/\mu$. The dynamics of such a theory is known, either in terms of dynamical superpotential with confined meson or baryon degrees of freedom [23], or in a dual description using magnetic degrees of freedom [4, 5]. By minimizing the potential of such low-energy effective actions we find the symmetry breaking pattern in each $N = 1$ vacuum. Supersymmetry and holomorphy imply that there are no phase transition at finite $|\mu|$: the qualitative features found here, such as the unbroken symmetry and the number of the Nambu-Goldstone bosons, are valid also at small nonvanishing μ .

This method not only allows us to cross-check the counting of the number of vacua in the previous section, but also enables us to determine the pattern of dynamical symmetry breaking from the first principles, as the limit of massless quarks can be studied exactly.

To be precise, we shall be interested here in the limit $m_i \rightarrow 0$ with μ and Λ ($\mu \gg \Lambda$) fixed. As long as μ is kept fixed, the limit $m_i \rightarrow 0$ is smooth and it is possible to determine how the exact flavor symmetry is realized in each vacuum.

This is to be contrasted to another limit, $\mu \rightarrow \infty$ with m_i fixed, which will be discussed in the next section. This latter limit, which does not commute with the former ($m_i \rightarrow 0$ first), is the relevant one for studying the decoupling the adjoint field and verifying the consistency with the known results in the standard $N = 1$ theories.

The analysis of this section will be divided in two parts, for small and for large values of n_f : this is necessary due to the emergence of the dual gauge group in the corresponding $N = 1$ theories for relatively large values of the flavor ($n_f > n_c + 1$ in $SU(n_c)$, $n_f > n_c + 2$ in $USp(2n_c)$).

For the ease of reading, a short summary is given at the end of this section: the reader who is more interested in the physics results than the technical aspects of the analysis, might well jump to it.

4.1 $SU(n_c)$ theories: Small Numbers ($n_f \leq n_c$) of Flavor

Let us first consider the cases, $n_f < n_c$. At large μ one can integrate over Φ :

$$\Phi^A = -\frac{\sqrt{2}}{\mu} \tilde{Q}_i^a (\lambda^A)_a^b Q_b^i; \quad (4.1)$$

resubstituting it into the superpotential one gets

$$W = -\frac{1}{2\mu} \left[\text{Tr} M^2 - \frac{1}{n_c} (\text{Tr} M)^2 \right] + \text{Tr}(Mm) + (n_c - n_f) \frac{\Lambda_1^{(3n_c - n_f)/(n_c - n_f)}}{(\det M)^{1/(n_c - n_f)}}, \quad (4.2)$$

where $M_i^j \equiv \tilde{Q}_i^a Q_a^j$ and where the known instanton-induced effective superpotential of $N = 1$ SQCD has been added.

$$\Lambda_1 \equiv \mu^{\frac{n_c}{3n_c - n_f}} \Lambda^{\frac{2n_c - n_f}{3n_c - n_f}} \quad (4.3)$$

is the scale of the $N = 1$ SQCD.

By making an ansatz for M :

$$M = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_f}), \quad (4.4)$$

the equations for λ_i are:

$$-\frac{1}{\mu} \left(\lambda_i - \frac{1}{n_c} \sum_j \lambda_j \right) + m_i - \frac{\Lambda_1^{(3n_c - n_f)/(n_c - n_f)}}{(\prod_j \lambda_j)^{1/(n_c - n_f)}} \lambda_i^{-1} = 0. \quad (4.5)$$

We now study the solutions of these equations in the limit $m_i \rightarrow 0$, with Λ and μ ($\mu \gg \Lambda$) fixed. By making an ansatz,

$$M = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_f}), \quad (4.6)$$

and upon multiplication with λ_i one finds (for each i)

$$-\frac{1}{\mu}(\lambda_i^2 - \lambda_i Y) + m_i \lambda_i + X = 0. \quad (4.7)$$

where

$$X \equiv -\Lambda_1^{(3n_c - n_f)/(n_c - n_f)} (\prod_j \lambda_j)^{\frac{1}{n_f - n_c}}, \quad Y \equiv \frac{1}{n_c} \sum_j^{n_f} \lambda_j. \quad (4.8)$$

One can take the limit $m_i \rightarrow 0$ directly in Eq.(4.7), which becomes simply

$$\lambda_i^2 - Y \lambda_i - \mu X = 0. \quad (4.9)$$

It can be solved by first assuming that X and Y are given:

$$\lambda_i = \frac{1}{2}(Y \pm \sqrt{Y^2 + 4\mu X}). \quad (4.10)$$

In general, r of λ_i 's can take the upper sign and the rest $(n_f - r)$ of λ_i 's the lower sign, $r = 0, 1, 2, \dots, n_f$. These solutions

$$\begin{aligned} \lambda_1 = \dots = \lambda_r &= \frac{1}{2}(Y + \sqrt{Y^2 + 4\mu X}); \\ \lambda_{r+1} = \dots = \lambda_{n_f} &= \frac{1}{2}(Y - \sqrt{Y^2 + 4\mu X}), \end{aligned} \quad (4.11)$$

must then be re-inserted into the definitions of X and Y to determine the latter quantities. One finds two relations

$$(2r - n_f)\sqrt{Y^2 + 4\mu X} = (2n_c - n_f)Y; \quad (4.12)$$

and

$$X = -\frac{\Lambda_1^{(3n_c - n_f)/(n_c - n_f)}}{2^{n_f/(n_f - n_c)}} \left(Y + \sqrt{Y^2 + 4\mu X} \right)^{\frac{r}{n_f - n_c}} \left(Y - \sqrt{Y^2 + 4\mu X} \right)^{\frac{n_f - r}{n_f - n_c}}. \quad (4.13)$$

These two equations can be easily solved for X and Y and give

$$Y = C \frac{\Lambda_1^{(3n_c - n_f)/(2n_c - n_f)}}{\mu^{(n_f - n_c)/(2n_c - n_f)}} e^{2\pi k i / (2n_c - n_f)} = C \Lambda \mu e^{2\pi k i / (2n_c - n_f)}, \quad k = 1, 2, \dots, 2n_c - n_f; \quad (4.14)$$

$$X = \frac{C'}{\mu} Y^2. \quad (4.15)$$

where C, C' are constants depending on n_f, n_c and r . Note that X is uniquely determined in terms of Y . At given r , then, there are $2n_c - n_f$ solutions for (X, Y) hence for $\{\lambda_i\}$. By summing over r , taking into account $n_f C_r$ ways of distributing r solutions with positive sign among n_f flavor, one appears to end up with $(2n_c - n_f) \sum_{r=0}^{n_f} n_f C_r = (2n_c - n_f) \cdot 2^{n_f}$ vacua.

Actually, one counts exactly twice each vacuum this way. The solution for $\{\lambda_i\}$'s depends on the value of r in a non trivial manner, in general. When r is replaced by $n_f - r$, however, C remains unchanged: the net change of (X, Y) and hence of $\{\lambda_i\}$'s, is that the two types of roots Eq.(4.10) are precisely interchanged, as can be seen from Eq.(4.12), Eq.(4.13), giving the same set of $\{\lambda_i\}$'s.

We find therefore

$$\mathcal{N}_1 = (2n_c - n_f) \cdot 2^{n_f - 1} \quad (4.16)$$

solutions of this type (i.e., finite in the $m_i \rightarrow 0$ limit). For $n_f < n_c$ these exhaust all possible solutions (see Eq.(3.24), Eq.(5.11)). They are classified by the value of an integer r : in a vacuum characterized by r , $U(n_f)$ symmetry of the theory is broken spontaneously to $U(r) \times U(n_f - r)$ by the condensates, Eq.(4.11).

The analysis for the case of $n_f = n_c$ is similar but can be made by using the superpotential valid for $n_c = n_f$ [23]

$$W = -\frac{1}{2\mu} \left[\text{Tr} M^2 - \frac{1}{n_c} (\text{Tr} M)^2 \right] + \text{Tr}(Mm) + \text{Tr} \kappa \{ (\det M) - B\tilde{B} - \Lambda^{2n_c} \}, \quad (4.17)$$

where B and \tilde{B} are baryonlike composite, $B = \epsilon_{i_1 i_2 \dots i_{n_c}} \epsilon^{a_1 a_2 \dots a_{n_c}} Q_{a_1}^{i_1} Q_{a_2}^{i_2} \dots Q_{a_{n_c}}^{i_{n_c}}$ and analogously for \tilde{B} in terms of \tilde{Q} 's, and κ is a Lagrange multiplier.

4.2 $SU(n_c)$: $n_f = n_c + 1$

In the case with $n_f = n_c + 1$ the effective superpotential is

$$\begin{aligned} W &= -\frac{1}{2\mu} \left[\text{Tr} M^2 - \frac{1}{n_c} (\text{Tr} M)^2 \right] + \text{Tr}(Mm) \\ &+ \frac{1}{\Lambda_1^{2n_f - 3}} \{ (\det M) - B_i (M)_j^i \tilde{B}^j \}, \end{aligned} \quad (4.18)$$

where

$$B_i = \epsilon_{i i_1 i_2 \dots i_{n_c}} \epsilon^{a_1 a_2 \dots a_{n_c}} Q_{a_1}^{i_1} Q_{a_2}^{i_2} \dots Q_{a_{n_c}}^{i_{n_c}}. \quad (4.19)$$

Set

$$M = \text{diag}(\lambda_1, \dots, \lambda_{n_f}), \quad (4.20)$$

$$\begin{aligned}
W &= -\frac{1}{2\mu} \left[\sum_i \lambda_i^2 - \frac{1}{n_c} \left(\sum_i \lambda_i \right)^2 \right] + \sum_i m_i \lambda_i \\
&+ \frac{1}{\Lambda_1^{2n_f-3}} \{ (\prod_j \lambda_j) - B_i \tilde{B}^i \lambda_i \}.
\end{aligned} \tag{4.21}$$

Derivation with respect to B_j and \tilde{B}^i yields

$$\lambda_i \tilde{B}^i = 0; \quad \lambda_i B_i = 0, \tag{4.22}$$

while derivation with respect to λ_i leads to

$$-\frac{1}{\mu} \left[\lambda_i - \frac{1}{n_c} \sum_j \lambda_j \right] + m_i + \frac{1}{\Lambda_1^{2n_f-3}} \{ \prod_{j \neq i} \lambda_j - B_i \tilde{B}^i \} = 0. \tag{4.23}$$

Multiplying with λ_i one gets (still no sum over i)

$$-\frac{1}{\mu} \left[\lambda_i^2 - \frac{1}{n_c} \lambda_i \sum_j \lambda_j \right] + m_i \lambda_i + \frac{1}{\Lambda_1^{2n_f-3}} \prod_j \lambda_j = 0. \tag{4.24}$$

Set now $Y = \frac{1}{n_c} (\sum_j \lambda_j)$, and $X = \prod_j \lambda_j$, and set $\Lambda_1 \equiv 1$ from now on. λ_i satisfies

$$\lambda_i^2 - (Y + m_i \mu) \lambda_i - \mu X = 0. \tag{4.25}$$

The solution is

$$\lambda_i = \frac{1}{2} (Y + m_i \mu \pm \sqrt{(Y + m_i \mu)^2 - 4\mu X}). \tag{4.26}$$

Note that when $m_i = 0$, $\lambda = B = 0$, all i 's, is a solution. We now classify the solutions according to the number (r) of λ 's which remain finite in the limit $m_i = 0$.

i) First consider the solution $\lambda_i \neq 0, \forall i$. In this case $B_i = \tilde{B}^i = 0, \forall i$. λ_i are given by

$$\lambda_i = \frac{1}{2} (Y \pm \sqrt{Y^2 - 4\mu X}). \tag{4.27}$$

The rest of the argument is the same as the one given Sec. 4.1 so one has $(2n_c - n_f) \cdot 2^{n_f-1}$ vacua of this kind.

ii) Consider now $\lambda_1 = 0, \lambda_j \neq 0, j \neq 1$. One finds that

$$B_j = \tilde{B}^j = 0, \quad j \neq 1; \tag{4.28}$$

$$\frac{1}{n_c} \sum_{j \neq 1} \lambda_j - \frac{\mu}{\Lambda_1^{2n_f-3}} B_1 \tilde{B}^1 = 0; \tag{4.29}$$

$$\lambda_j - \frac{1}{n_c} \sum_{k \neq 1} \lambda_k - m_j \mu = 0. \tag{4.30}$$

The last equation has the form,

$$C \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{n_f} \end{pmatrix} - \mu \begin{pmatrix} m_2 \\ m_3 \\ \vdots \\ m_{n_f} \end{pmatrix} = 0, \quad (4.31)$$

with

$$\det C = 0. \quad (4.32)$$

For generic m_i 's therefore this equation has no solutions. Here it is important that we consider the massless limit of massive theory, not directly massless theory.

iii) Consider now $\lambda_i = 0$, $i = 1, 2, \dots, r$, and $\lambda_j \neq 0$, $j \geq r + 1$. One gets

$$B_j = \tilde{B}^j = 0, \quad j = r + 1, \dots, n_f \quad (4.33)$$

and

$$\frac{1}{n_c} \sum_{j>r} \lambda_j - \frac{\mu}{\Lambda^{2n_f-3}} B_i \tilde{B}^i = 0, \quad i = 1, 2, \dots, r; \quad (4.34)$$

$$\lambda_j - \frac{1}{n_c} \sum_{k>r} \lambda_k - m_j \mu = 0, \quad j = r + 1, \dots, n_f. \quad (4.35)$$

The last equations now give finite answers for λ_j : but they are of $O(m_i)$ and approach 0 as $m_i \rightarrow 0$. One gets then also,

$$B_i, \tilde{B}^i \rightarrow 0, \quad m_i \rightarrow 0. \quad (4.36)$$

Therefore all these cases degenerate into one single solution,

$$\lambda = B = \tilde{B} = 0. \quad (4.37)$$

To conclude, we find in the massless limit $(2n_c - n_f) \cdot 2^{n_f-1}$ solutions with finite λ 's and one solution with vanishing vevs. The latter is consistent with the general formula Eq.(4.47) for the second group of vacua (with vanishing VEVS) found for $n_f > n_c + 1$ below, since $\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c-1} n_f C_r (\tilde{n}_c - r) = 1$, for $n_f = n_c + 1$.

4.3 $SU(n_c)$: large numbers ($n_f > n_c + 1$) of flavor

In the cases $n_f > n_c + 1$, the effective low energy degrees of freedom are dual quarks and mesons [4]. The effective superpotential is given by

$$\mathcal{W} = \tilde{q} M q + \text{Tr}(m M) - \frac{1}{2\mu} \left[\text{Tr} M^2 - \frac{1}{n_c} (\text{Tr} M)^2 \right], \quad (4.38)$$

where q 's are n_f sets of dual quarks in the fundamental representation of the dual gauge group $SU(\tilde{n}_c)$, with $\tilde{n}_c = n_f - n_c$. The vacuum equations following from Eq. (4.38) are:

$$M_{ij} q_j^\alpha = 0; \quad \tilde{q}_\alpha^i M_{ij} = 0; \quad (4.39)$$

$$\tilde{q}_\alpha^i q_j^\alpha + \delta_{ij} m_i - \frac{1}{\mu} \left(M_{ij} - \frac{1}{n_c} (\text{Tr} M) \delta_{ij} \right) = 0. \quad (4.40)$$

The first set of equations tell us that the meson matrix M and the dual squarks are orthogonal in the flavor space. By using the dual color and flavor rotations (and the use of Eq. (4.40)) the dual squarks can be taken to be nonvanishing in the first r flavors and of the form:

$$q_i^\alpha = \begin{pmatrix} d_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ d_r \\ 0 \\ \vdots \end{pmatrix}; \quad q_i^\alpha = 0, \quad i = r+1, \dots, n_f. \quad (4.41)$$

$$\tilde{q}_\alpha^i = \begin{pmatrix} \tilde{d}_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{d}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ \tilde{d}_r \\ 0 \\ \vdots \end{pmatrix}; \quad \tilde{q}_\alpha^i = 0, \quad i = r+1, \dots, n_f, \quad (4.42)$$

where⁶

$$r = 0, 1, 2, \dots, \tilde{n}_c - 1, \quad (4.43)$$

and

$$d_i \tilde{d}_i + m_i + \frac{1}{\mu n_c} \cdot \text{Tr} M = 0. \quad (4.44)$$

Eq. (4.40) implies also that the meson matrix is diagonal,

$$M = \text{diag}(0, 0, \dots, 0, \lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_{n_f}), \quad (4.45)$$

$$\lambda_i - \frac{1}{n_c} \sum \lambda = m_i \mu. \quad (4.46)$$

Clearly the last equations determine uniquely all λ 's: on the other hand there are ${}_f C_r$ choices of masses which enter the equations for nonzero squark VEVs. Furthermore, because the vacua with r nonzero entries in the squark vevs leave an $SU(\tilde{n}_c - r)$ dual gauge group unbroken (M being singlet), each such vacuum must be counted as $\tilde{n}_c - r$ vacua (Witten's index). In all, then, there are

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c-1} {}_f C_r (\tilde{n}_c - r) \quad (4.47)$$

vacua, with $\text{Rank } M < n_f$, in which the global $U(n_f)$ symmetry remains unbroken, in the $m_i \rightarrow 0$ limit.

We seem to face a difficulty, however. The number of vacua found here is less than the known total number of vacua \mathcal{N} (Eq.(3.23)). Where are other vacua?

⁶ Note that the value $r = \tilde{n}_c$ should be excluded. In this case, $n_f - r = n_c$ and the nonvanishing meson submatrix is $n_c \times n_c$. Eq. (4.40) for $i, j = r+1, \dots, n_f$ have no solutions since the matrix $M_{ij} - \frac{1}{n_c} (\text{Tr} M) \delta_{ij}$ is of rank $n_c - 1$ while $\delta_{ij} m_i$ has rank n_c .

This apparent puzzle can be solved once the nontrivial $SU(\tilde{n}_c)$ dynamics are taken into account⁷. If the VEVs of the mesons have Rank $M = n_f$, dual quarks are all massive. The theory becomes pure Yang-Mills type in the infrared, and the strong interaction effects of dual gauge dynamics must be properly taken into account. By integrating out the dual quarks out, we find the effective superpotential,

$$W_{eff} = -\frac{1}{2\mu} \left[\text{Tr} M^2 - \frac{1}{n_c} (\text{Tr} M)^2 \right] + \text{Tr}(Mm) + \Lambda_1^{(3n_c - n_f)/(n_c - n_f)} (\det M)^{1/(n_f - n_c)}. \quad (4.48)$$

The analysis of the vacua of this effective action is similar to that of the $n_f < n_c$ cases, Eqs.(4.5)–(4.15): the superpotential (4.48) is actually identical to (more precisely, continuation of) (4.2)!⁸ One finds therefore $\mathcal{N}_1 = (2n_c - n_f) \cdot 2^{n_f - 1}$ solutions with finite VEVs in exactly the same way as in Eqs.(4.5)–(4.15). In these vacua, classified by an integer r , $U(n_f)$ symmetry of the theory is broken spontaneously to various $U(r) \times U(n_f - r)$.

It is now possible to make a highly nontrivial consistency test, by the vacuum counting. By using $\tilde{n}_c = n_f - n_c$ and changing the summation index from r to $n_f - r$, it is easy to show that

$$\begin{aligned} \mathcal{N}_2 &= \sum_{r=0}^{n_f} n_f C_r (r - n_c) - \sum_{r=0}^{n_c} n_f C_r (r - n_c) \\ &= -(2n_c - n_f) \cdot 2^{n_f - 1} + \sum_{r=0}^{n_c - 1} n_f C_r (n_c - r). \end{aligned} \quad (4.49)$$

The total number of the $N = 1$ vacua is therefore found to be

$$\mathcal{N}_1 + \mathcal{N}_2 = \sum_{r=0}^{n_c - 1} n_f C_r (n_c - r), \quad (4.50)$$

which is the correct answer (see Eq. (3.23)) for $n_f > n_c + 1$!

4.4 $USp(2n_c)$ Theories: Small Numbers ($n_f \leq n_c + 1$) of Flavor

We start with the cases $n_f \leq n_c + 1$. At large μ the equation of motion for the Φ superfield is:

$$\Phi^A = -\frac{1}{\sqrt{2}\mu} (Q_\alpha^i S_{\alpha\beta}^A Q_\beta^i), \quad (4.51)$$

where $\Phi = \Phi^A S^A$ and S^A are the $USp(2n_c)$ generators⁹. Resubstituting it into the superpotential (3.25), and accounting for the instanton-induced contribution for $n_f \leq n_c$, one gets:

$$W = -\frac{1}{8\mu} \text{Tr} M J M J - \frac{1}{2} \text{Tr} m M + (n_c + 1 - n_f) \frac{\Lambda_1^{3+2n_f/(n_c+1-n_f)}}{(\text{Pf} M)^{1/(n_c+1-n_f)}}, \quad (4.52)$$

⁷ In fact, a related puzzle is how Seiberg's dual Lagrangian [4] - the first two terms of Eq. (4.38) - can give rise to the right number of vacua for the massive $N = 1$ SQCD with $n_f > n_c + 1$. By following the same method as below but with $\mu = \infty$, we do find the correct number (n_c) of vacua.

⁸ In the case of SQCD ($\mu = \infty$) this observation is in agreement with the well known fact that, in spite of distinct physical features at $n_f \leq n_c$ and $n_f \geq n_c + 1$, the results for the squark and gaugino condensates, $\langle m_i \tilde{Q}^i Q^i \rangle = \langle \lambda \lambda \rangle = (\det m)^{1/n_c} \Lambda_1^{3-n_f/n_c}$ hold true for all values of the flavor and color as long as $n_f < 3n_c$ [24].

⁹ With respect to the \hat{S}^A generators defined in App. A, $S^A = \hat{S}^A J$ with $S_{\alpha\beta}^A = S_{\beta\alpha}^A$ and $\text{Tr} S^A J S^B J = \frac{1}{2} \delta^{AB}$.

where $M^{ij} = Q_a^i J^{ab} Q_b^j$'s are the meson-like composite superfields, $m = -i\sigma_2 \otimes \text{diag}(m_1, \dots, m_{n_f})$ is the mass matrix, and $\Lambda_1^{2(3n_c+3-n_f)} = \mu^{2n_c+2} \Lambda^{2(2n_c+2-n_f)}$.

With the ansatz $M = i\sigma_2 \otimes \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_f})$, then the superpotential is ($\Lambda_1 = 1$ below):

$$W = \frac{1}{4\mu} \sum_{i=1}^{n_f} \lambda_i^2 - \sum_{i=1}^{n_f} m_i \lambda_i + (n_c + 1 - n_f) \frac{1}{(\prod_{i=1}^{n_f} \lambda_i)^{1/(n_c+1-n_f)}}, \quad (4.53)$$

and the vacuum equations become:

$$\frac{1}{2\mu} \lambda_i - m_i - \frac{1}{\lambda_i} \frac{1}{(\prod_j \lambda_j)^{1/(n_c+1-n_f)}} = 0. \quad (4.54)$$

Since the last term is common to all i , we find

$$\lambda_i = \mu(m_i \pm \sqrt{m_i^2 + (2X/\mu)}), \quad X \equiv \frac{1}{(\prod_j \lambda_j)^{1/(n_c+1-n_f)}} \quad (4.55)$$

We choose r negative signs and $n_f - r$ positive signs in the roots of λ_i . In strictly massless limit $m = 0$, the definition of X gives¹⁰

$$X \equiv \frac{1}{(\prod_j \lambda_j)^{1/(n_c+1-n_f)}} = \frac{1}{((-1)^r (2X\mu)^{n_f/2})^{1/(n_c+1-n_f)}} \quad (4.56)$$

and hence

$$X_0 = \Lambda_1^{\frac{2(3n_c+3-n_f)}{2n_c+2-n_f}} (2\mu)^{\frac{-n_f}{2n_c+2-n_f}} e^{2\pi i k/(2n_c+2-n_f)}, \quad k = 1, 2, \dots, 2n_c + 2 - n_f, \quad (4.57)$$

where the subscript 0 indicates that this is for the massless quark limit and we have reinstated the dependence on the scale Λ_1 . Note that

$$X_0 \propto \mu \Lambda^2 \quad (4.58)$$

in terms of the $N = 2$ scale factor Λ and μ . There are $(2n_c + 2 - n_f)$ roots for X_0 . Up to $O(m)$, we find

$$X = X_0 \left[1 + \frac{2(\sum_{j=1}^r m_j - \sum_{j=r+1}^{n_f} m_j)}{2n_c + 2 - n_f} \left(\frac{\mu}{2X_0} \right)^{1/2} \right], \quad (4.59)$$

and

$$\lambda_i = \pm \mu \left[\left(\frac{2X_0}{\mu} \right)^{1/2} + \frac{\sum_{j=1}^r m_j - \sum_{j=r+1}^{n_f} m_j}{2n_c + 2 - n_f} \pm m_i \right] \quad (4.60)$$

There appears to be 2^{n_f} choices for the signs among λ_i 's; actually, the signs of λ_i must be such that Eq. (4.56), and not its $2(n_c + 1 - n_f)$ th power, is satisfied. This restricts the choices of the signs by half: for a particular phase of X_0 with k even or odd, the number of minus signs among λ_i must be even or odd, respectively. In total, there are

$$(2n_c + 2 - n_f) 2^{n_f-1} \quad (4.61)$$

¹⁰ Eq. (4.55) and analogous relations in $SU(n_c)$ case clearly show the non commutativity of the two limits, $\mu \rightarrow 0$ first with Λ_1, m_i fixed to be studied in the next section, and $m_i \rightarrow 0$ first with $\mu \gg \Lambda$ fixed, being examined here.

vacua, which is consistent with the previous method as well as the semi-classical method in the previous section (Eq. (3.44)).

Since the massless limit gives $SO(2n_f)$ flavor symmetry, the choice of different signs in λ_i for each flavor strongly suggests the vacua to form a spinor representation of $SO(2n_f)$. The constraint that the number of minus signs to be even or odd implies that it is a spinor representation of a definite chirality. In fact, all of these vacua transform among each other under $SO(2n_f)$ group because it can flip the signs of two eigenvalues at the same time, consistent with an irreducible representation of $SO(2n_f)$. The equal mass case $m_i = m$ has $U(n_f)$ flavor symmetry and the vacua above form $n_f C_r$ multiplets (with even or odd r), consistent with the decomposition of the $SO(2n_f)$ spinor to even or odd-rank r anti-symmetric tensors under $U(n_f)$. In each group of vacua, the global symmetry is broken to $U(r) \times U(n_f - r)$.

One of the most important results of this section is that the meson condensates in the massless limit always break $SO(2n_f)$ flavor symmetry as

$$SO(2n_f) \rightarrow U(n_f), \quad (4.62)$$

matching nicely with the expectation discussed in Section 2.

When $n_f = n_c + 1$, the large μ theory develops a quantum modified constraint

$$W = -\frac{1}{8\mu} \text{Tr} M^2 - \frac{1}{2} \text{Tr} m M + X(\text{Pf} M - \Lambda_1^{2n_f}). \quad (4.63)$$

Following the similar analysis as above, we again find the total number of vacua to be $(2n_c + 2 - n_f) 2^{n_f-1}$, consistent with the semi-classical method.

When $n_f = n_c + 2$, the large μ theory develops a superpotential

$$W = -\frac{1}{8\mu} \text{Tr} M^2 - \frac{1}{2} \text{Tr} m M + \frac{\text{Pf} M}{\Lambda_1^{2n_f-3}}. \quad (4.64)$$

Following the similar analysis as above, we again find the total number of vacua to be $n_c 2^{n_f-1} + 1$, consistent with the semi-classical method. The last vacuum corresponds to the case without symmetry breaking $\lambda_i = 2m_i \mu \rightarrow 0$ in the massless quark limit.

4.5 $USp(2n_c)$ Theories: Large Numbers ($n_f > n_c + 2$) of Flavor

Next consider the cases $n_f > n_c + 2$. The large μ theory has a description in terms of the dual magnetic gauge group $USp(2\tilde{n}_c) = USp(2(n_f - n_c - 2))$ and magnetic quarks q ,

$$W = -\frac{1}{8\mu} \text{Tr} M^2 - \frac{1}{2} \text{Tr} m M + \frac{1}{\mu_m} q M q, \quad (4.65)$$

where the scale μ_m is the matching scale between the electric and magnetic gauge couplings. As in the $SU(n_c)$ cases above, it is only consistent to use this effective action to get information on the vacuum properties as long as dual quarks turn out to be light. Otherwise, the nontrivial dual gauge dynamics must be taken into account.

By minimizing the potential of the magnetic theory with dual quarks directly, we find vacua characterized by VEVs

$$q_i = \sqrt{\mu m_i}, \quad \lambda_i = 0, \quad (4.66)$$

for r flavors and

$$q_i = 0, \quad \lambda_i = m_i \mu, \quad (4.67)$$

for the remaining $n_f - r$ flavors. r is restricted to be $r \leq \tilde{n}_c = n_f - n_c - 2$. For each value of r , there are ${}_{n_f}C_r$ choices on which flavor to have non-vanishing q_i , and there is unbroken $USp(2(n_f - n_c - 2 - r))$ gauge group. It develops the gaugino condensate, giving $n_f - n_c - 1 - r$ vacua each. The number of vacua of this type is

$$\mathcal{N}_2 = \sum_{r=0}^{n_f - n_c - 2} {}_{n_f}C_r (n_f - n_c - 1 - r). \quad (4.68)$$

Changing the variable r to $n_f - r$, it is rewritten as

$$\mathcal{N}_2 = \sum_{r=n_c+2}^{n_f} {}_{n_f}C_r (r - n_c - 1) = \sum_{r=0}^{n_f} {}_{n_f}C_r (r - n_c - 1) + \sum_{r=0}^{n_c+1} {}_{n_f}C_r (n_c + 1 - r). \quad (4.69)$$

The first sum can be computed and gives

$$\mathcal{N}_2 = -(2n_c + 2 - n_f) 2^{n_f-1} + \sum_{r=0}^{n_c} {}_{n_f}C_r (n_c + 1 - r). \quad (4.70)$$

Note that the sum in the second term can be stopped at $r = n_c$ because the argument vanishes for $r = n_c + 1$. Therefore the total $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ agrees with the counting of classical vacua. On the other hand, this gives a nice interpretation of the number that the “extra” contribution \mathcal{N}_2 signals the emergence of the dual gauge group in the massless quark limit. In this group of vacua (present only for larger values of n_f), the chiral symmetry is not spontaneously broken in the limit, $m_i \rightarrow 0$.

In order to get the vacua with $\text{Rank} M = n_f$, one must integrate out the dual quarks first and consider the resulting effective action:

$$W_{\text{eff}} = -\frac{1}{8\mu} \text{Tr} M^2 - \frac{1}{2} \text{Tr} m M + (n_c + 1 - n_f) \left[\text{Pf} \frac{M}{\mu_m} \tilde{\Lambda}^{3(n_f - n_c - 1) - n_f} \right]^{1/(n_f - n_c - 1)}. \quad (4.71)$$

By making the ansatz, $M = i\sigma_2 \otimes \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_f})$, the vacuum equation becomes

$$\left(\prod_i^{n_f} \frac{\lambda_i}{\mu_m} \right)^{1/(n_f - n_c - 1)} \tilde{\Lambda}^{(3(n_f - n_c - 1) - n_f)/(n_f - n_c - 1)} = m_i \lambda_i - \frac{\lambda_i^2}{\mu}. \quad (4.72)$$

Call the right hand side, which is flavor independent, X . We have two solutions for λ_i for given X ,

$$\lambda_i = \frac{\mu}{2} \left[m_i \pm \sqrt{m_i^2 + 4X/\mu} \right]. \quad (4.73)$$

In the massless limit, $\lambda_i = \sqrt{X\mu}$, which in turns gives

$$X = (X\mu)^{n_f/2(n_f - n_c - 1)} \tilde{\Lambda}^{(3(n_f - n_c - 1) - n_f)/(n_f - n_c - 1)} \mu_m^{-n_f/(n_f - n_c - 1)}. \quad (4.74)$$

The solution is given by

$$X^{2n_c+2-n_f} = \tilde{\Lambda}^{-2(3(n_f-n_c-1)-n_f)} \mu_m^{2n_f} \mu^{-n_f}. \quad (4.75)$$

This obviously gives $2n_c+2-n_f$ solutions, for which there are 2^{n_f-1} possibilities on the sign choices for each λ_i . Therefore we find

$$\mathcal{N}_1 = (2n_c + 2 - n_f) 2^{n_f-1} \quad (4.76)$$

vacua. In this set of vacua X depends on $\tilde{\Lambda}$ and has a finite $m_i \rightarrow 0$ limit (*i.e.*, λ_i stay non-vanishing, justifying the assumption of the maximal rank meson matrix). Since all λ_i 's are equal in the magnitude in this limit, the chiral $SO(2n_f)$ symmetry is spontaneously broken as

$$SO(2n_f) \rightarrow U(n_f) \quad (4.77)$$

in all vacua belonging to this group. Note that this number of vacua would precisely corresponds to that of monopole condensation in the $SO(2n_f)$ spinor representation at the Chebyshev points of the curve (see below).

The total number of vacua at large μ found, $\mathcal{N}_1 + \mathcal{N}_2 = \sum_{r=0}^{n_c} n_f C_r (n_c + 1 - r)$, agrees with that of the semiclassical theories.

4.6 Summary of Section 4

The number of $N = 1$ vacua and the pattern of symmetry breaking in each of them has thus been determined in $SU(n_c)$ and $USp(2n_c)$ theories at large μ , from the first principles. For small numbers of flavor ($n_f \leq n_c + 1$ for $SU(n_c)$; $n_f \leq n_c + 2$ for $USp(2n_c)$) the low-energy degrees of freedom are meson-like (sometimes also baryon-like) composites: their condensation lead to a definite pattern of symmetry breaking in each vacuum.

$SU(n_c)$ theories have an exact global $U(n_f)$ symmetry in the equal mass (or massless) limit, which is spontaneously broken to $U(r) \times U(n_f - r)$ in $(n_c - r) n_f C_r$ vacua, $r = 0, 1, \dots, [n_f/2]$. The number of the vacua with the particular pattern of symmetry breaking will match exactly with those found from the analysis of low-energy monopole/dual quark effective action, to be analyzed in the next sections.

In $USp(2n_c)$ theories, for small numbers of flavor, *the chiral symmetry ($SO(2n_f)$) in the massless limit is always spontaneously broken down to an unbroken $U(n_f)$* . This result nicely agrees with what is expected generally from bifermion condensates of the standard form in non supersymmetric theories. This fact that various vacua have exactly the same symmetry breaking pattern, has an important consequence in the physics at small μ , to be studied below.

The difference in the symmetry breaking pattern in $SU(n_c)$ and $USp(2n_c)$ theories reflects the structures of the low-energy effective actions of the respective theories, which in turn is a direct consequence of the different structure of the two types of gauge groups, see Eq.(4.2) and Eq.(4.52).

For larger numbers of flavor ($n_f > n_c + 1$ for $SU(n_c)$; $n_f > n_c + 2$ for $USp(2n_c)$) the low-energy degrees of freedom are dual quarks and gluons, as well as some mesons. In these cases, besides

the vacua with the properties mentioned above, other vacua exist in which all VEVs vanish and in which the global symmetry ($SU(n_f)$ for $SU(n_c)$; $SO(2n_f)$ in $USp(2n_c)$ theories) remains unbroken in the massless limit ($m_i \rightarrow 0, \forall i$).

5 Decoupling of the Adjoint Fields

In this section we discuss briefly another limit, in which μ is sent to ∞ , keeping m_i and $\Lambda_1 \equiv \mu^{\frac{n_c}{3n_c-n_f}} \Lambda^{\frac{2n_c-n_f}{3n_c-n_f}}$ fixed. We first re-analyse the number of the $N = 1$ vacua in the regime, $\mu \gg \Lambda_1, m_i$, and reproduce the correct multiplicity of vacua. Since the two limits ($\mu \rightarrow \infty$ first and $m_i \rightarrow 0$ first) do not commute, this provides for an independent check of the vacuum counting. Subsequently, taking the decoupling limit, $\mu \rightarrow \infty$, we identify the standard supersymmetric vacua of the theories without the adjoint field Φ .

5.1 $SU(n_c)$

First consider the cases with small number of flavors, $n_f \leq n_c$, and go back to the equations for λ_i

$$-\frac{1}{\mu} \left(\lambda_i - \frac{1}{n_c} \sum_j \lambda_j \right) + m_i - \frac{1}{n_c - n_f} \frac{\Lambda_1^{(3n_c-n_f)/(n_c-n_f)}}{(\prod_j \lambda_j)^{1/(n_c-n_f)}} \lambda_i^{-1} = 0. \quad (5.1)$$

following from

$$W = -\frac{1}{2\mu} \left[\text{Tr} M^2 - \frac{1}{n_c} (\text{Tr} M)^2 \right] + \text{Tr}(Mm) + \frac{\Lambda_1^{(3n_c-n_f)/(n_c-n_f)}}{(\det M)^{1/(n_c-n_f)}}, \quad (5.2)$$

where $M_i^j \equiv \tilde{Q}_i^a Q_a^j$; $\langle M_i^j \rangle = \text{diag}(\lambda_1, \dots, \lambda_{n_f})$. The scale of the $N = 1$ SQCD

$$\Lambda_1 \equiv \mu^{\frac{n_c}{3n_c-n_f}} \Lambda^{\frac{2n_c-n_f}{3n_c-n_f}} \quad (5.3)$$

must be kept fixed in the $\mu \rightarrow \infty$ limit, to recover the standard $N = 1$ SQCD.

In the large μ limit, some of the λ_i 's of Eq.(5.1) are of the order of μ , while others are much smaller. The solutions can thus be classified according to the number r of the λ_i 's which are of the order of μ . The large λ_i 's (say $\lambda_1, \lambda_2, \dots, \lambda_r$) satisfy (setting $\Lambda_1 = 1$ from now on)

$$\lambda_i - \frac{1}{n_c} \sum_{k=1}^r \lambda_k - \mu m_i \simeq 0; \quad (5.4)$$

which nicely corresponds to Eq. (3.19). The justification for dropping the last term of Eq. (5.1) will be shortly given.

The smaller eigenvalues λ_p can be found as follows: substituting the approximate solutions for the large λ_i 's (see Eq. (3.21))

$$\lambda_i \simeq \mu m_i + \frac{1}{n_c - r} \mu \sum_{k=1}^r m_k = (m_i + c) \mu \quad (5.5)$$

into Eq. (5.1) with $p = r + 1, r + 2, \dots, n_f$, one finds

$$(c + m_p) \lambda_p = \frac{1}{n_c - n_f} \frac{1}{(\prod_{i=1}^r \lambda_i \cdot \prod_{q=r+1}^{n_f} \lambda_q)^{1/n_c-n_f}}. \quad (5.6)$$

This can be further put in the form,

$$(\lambda_p)^{n_c - n_f} \prod_q \lambda_q = \frac{A_p}{\prod_{i=1}^r \lambda_i} = \frac{B_p}{\mu^r}, \quad (5.7)$$

where A_p and B_p are some finite constants depending on the masses. By taking the product over all $p = r + 1, r + 2, \dots, n_f$, one gets

$$\left(\prod_q \lambda_q\right)^{n_c - r} = \frac{\text{const.}}{\mu^{r(n_f - r)}}, \quad (5.8)$$

therefore

$$\prod_q \lambda_q = \frac{\text{const.}}{\mu^{r(n_f - r)/(n_c - r)}} \cdot \exp 2\pi i k / (n_c - r), \quad (k = 1, 2, \dots, n_c - r). \quad (5.9)$$

Once $\prod_q \lambda_q$ is determined, each of the small eigenvalues can be found uniquely from Eq. (5.6), so that each choice of r large λ_i 's yields $n_c - r$ solutions.

It is easy to see from Eq. (5.9) that the last term of Eq. (5.1) behaves as

$$\mu^{-n_c / (n_c - r)} \quad (5.10)$$

and thus is indeed negligible as compared to the terms kept in Eq. (5.4), as long as $r < n_c$.

The total number of the vacua at large μ is thus

$$\mathcal{N} = \sum_{r=0}^{n_f} (n_c - r) \binom{n_f}{r} \quad (5.11)$$

which coincides with the number of the classical vacua, Eq. (3.23).

With $n_f > n_c + 1$, a naïve use of Eq.(4.38) in the limit $\mu \rightarrow \infty$ would lead to no supersymmetric vacua. The correct vacua can be found by taking into account the nontrivial dual gauge dynamics and consequently considering the effective action Eq.(4.48): the analysis of the decoupling limit is then similar to the $n_f \leq n_c$ cases discussed above. A subtle new point however is that now the number of “large” eigenvalues r , can *a priori* exceed n_c . Actually, however, for these values of r the last term of Eq. (5.1) becomes dominant and invalidates the solution (see Eq.(5.10)): the sum over r must be truncated at $r = n_c - 1$.¹¹ One thus finds for the number of vacua

$$\mathcal{N} = \sum_{r=0}^{n_c - 1} (n_c - r) \binom{n_f}{r} \quad \text{for} \quad n_f > n_c + 1, \quad (5.12)$$

which is indeed the correct vacuum multiplicity in this case (Eq. (3.23)).

In the $\mu \rightarrow \infty$ limit, all solutions except for those with $r = 0$ have some VEVs running away to infinity. They do not belong to the space of vacua of the $N = 1$ supersymmetric QCD. Only the $r = 0$ solutions are characterized by finite VEVs,

$$m_i \langle Q_i \tilde{Q}_i \rangle = \text{indep. of } i = \Lambda_1^{\frac{3n_c - n_f}{n_c}} \left(\prod_{j=1}^{n_f} m_j^{1/n_c} \right) \cdot e^{2\pi i k / n_c}, \quad k = 1, 2, \dots, n_c : \quad (5.13)$$

they are indeed the well-known n_c vacua of $N = 1$ SQCD [24].

¹¹From Eq.(5.4) it can be seen that the case $r = n_c$ is also excluded.

5.2 $USp(2n_c)$

For small numbers of flavors ($n_f \leq n_c$) the equations for λ_i 's are (Eq. (4.54)):

$$\frac{1}{2\mu}\lambda_i - m_i - \frac{1}{\lambda_i} \frac{1}{(\prod_j \lambda_j)^{1/(n_c+1-n_f)}} = 0. \quad (5.14)$$

The large λ_i 's (say $\lambda_1, \lambda_2, \dots, \lambda_r$) satisfy:

$$\lambda_i \simeq 2m_i\mu \quad i = 1, 2, \dots, r, \quad (5.15)$$

where the last term of Eq. (5.14) is negligible, as will be shown shortly. The $n_f - r$ smaller λ_p 's are found by substituting Eq. (5.15) into Eq. (5.14):

$$m_p\lambda_p = \frac{1}{(\prod_i \lambda_i \prod_q \lambda_q)^{1/(n_c+1-n_f)}}, \quad (5.16)$$

where $i = 1, 2, \dots, r$ runs over the large λ 's and $p, q = r+1, r+2, \dots, n_f$ refer to the smaller ones. This can be further rewritten as:

$$(\prod_p \lambda_p)^{n_c+1-r} = \frac{\text{const.}}{\mu^{r(n_f-r)}}, \quad (5.17)$$

and thus

$$\prod_p \lambda_p = \frac{\text{const.}}{\mu^{r(n_f-r)/(n_c+1-r)}} \exp 2\pi i k / (n_c + 1 - r) \quad \text{with } k = 1, 2, \dots, n_c + 1 - r. \quad (5.18)$$

One can see that each choice of r large λ_i 's yields $n_c + 1 - r$ solutions. It is easy to show that the last term of Eq. (5.14) is indeed negligible, behaving as

$$\mu^{-(n_c+1)/(n_c+1-r)} \quad (5.19)$$

as long as $r \leq n_c$ (see below).

Summing over r one finds for the total number of the vacua

$$\mathcal{N} = \sum_{r=0}^{n_f} (n_c + 1 - r) \binom{n_f}{r} \quad (5.20)$$

which agrees (for $n_f \leq n_c$) with the number of the classical vacua, Eq. (3.43); the above solutions therefore exhaust all possible vacua of the theory.

The analysis for the cases with larger number of flavor ($n_f > n_c + 1$) is quite similar to the one made above for smaller values of n_f . The only difference is that for $r > n_c$ it is no longer correct to neglect the last term of Eq. (5.14), as can be seen from Eq. (5.19), hence the sum over r must stop at $r = n_c$. The case $r = n_c + 1$ might look subtle, but it is clear from Eq. (5.17) that no solution exists in this case either. One thus ends up with the number of vacua,

$$\mathcal{N} = \sum_{r=0}^{n_c} (n_c - r + 1) \cdot \binom{n_f}{r}, \quad (5.21)$$

which is the correct result (see Eq. (3.43)).

Again, in the strict $\mu = \infty$ limit, only the vacua with finite VEVs must be retained. They are the solutions corresponding to $r = 0$ above: we find precisely $n_c - 1$ solutions of the $N = 1$ $USp(2n_c - 1)$ theory without the adjoint matter fields.

6 Microscopic Picture of Dynamical Symmetry Breaking

In this and in three subsequent sections we seek for a microscopic understanding of the mechanism of dynamical flavor symmetry breaking as well as of confinement itself, by studying these theories at small μ and small m_i . It is necessary to analyze the $N = 2$ vacua on the Coulomb branch which survive the $\mu \neq 0$ perturbation. The auxiliary genus $g = n_c - 1$ (or n_c) curves for $SU(n_c)$ ($USp(2n_c)$) theories are given by

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda^{2n_c - n_f} \prod_{j=1}^{n_f} (x + m_j), \quad SU(n_c), \quad n_f \leq 2n_c - 2, \quad (6.1)$$

and

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda \prod_{j=1}^{n_f} \left(x + m_j + \frac{\Lambda}{n_c} \right), \quad SU(n_c), \quad n_f = 2n_c - 1, \quad (6.2)$$

with ϕ_k subject to the constraint $\sum_{k=1}^{n_c} \phi_k = 0$, and

$$xy^2 = \left[x \prod_{a=1}^{n_c} (x - \phi_a^2)^2 + 2\Lambda^{2n_c + 2 - n_f} m_1 \cdots m_{n_f} \right]^2 - 4\Lambda^{2(2n_c + 2 - n_f)} \prod_{i=1}^{n_f} (x + m_i^2), \quad USp(2n_c). \quad (6.3)$$

The connection between these genus g hypertori and physics is made [1]-[3] through the identification of various period integrals of the holomorphic differentials on the curves with $(da_{Di}/du_j, da_i/du_j)$, where the gauge invariant parameters u_j 's are defined by the standard relation,

$$\prod_{a=1}^{n_c} (x - \phi_a) = \sum_{k=0}^{n_c} u_k x^{n_c - k}, \quad u_0 = 1, \quad u_1 = 0, \quad SU(n_c); \quad (6.4)$$

$$\prod_{a=1}^{n_c} (x - \phi_a^2) = \sum_{k=0}^{n_c} u_k x^{n_c - k}, \quad u_0 = 1, \quad USp(2n_c), \quad (6.5)$$

and $u_2 \equiv \langle \text{Tr} \Phi^2 \rangle$, etc. The VEVs of a_{Di} , a_i , which are directly related to the physical masses of the BPS particles through the exact Seiberg-Witten mass formula [1, 2]

$$M^{n_{mi}, n_{ei}, S_k} = \sqrt{2} \left| \sum_{i=1}^g (n_{mi} a_{Di} + n_{ei} a_i) + \sum_k S_k m_k \right|, \quad (6.6)$$

are constructed as integrals over the non-trivial cycles of the meromorphic differentials on the curves. See Appendix C.

We require that the curve is maximally singular, i.e. $g = n_c - 1$ (or n_c for $USp(2n_c)$) pairs of branch points to coincide: this determines the possible values of $\{\phi_a\}$'s. These points correspond to the $N = 1$ vacua, for the particular $N = 1$ perturbation, Eq.(1.2). Note that as we work with generic and nonvanishing quark masses (and then consider $m_i \rightarrow 0$ limit), this is an unambiguous procedure to identify all the $N = 1$ vacua of our interest.¹²

¹²There are other kinds of singularities of $N = 2$ QMS at which, for instance, three of the branch points meet. These correspond to $N = 1$ vacua, selected out by different types of perturbations such as $\text{Tr} \Phi^3$, which are not considered here.

In fact, near one of the singularities where dyons with quantum numbers

$$(n_{m1}, n_{m2}, \dots, n_{mg}; n_{e1}, n_{e2}, \dots, n_{eg}) = (1, 0, \dots, 0; 0, \dots), \dots, (0, 0, \dots, 1; 0, \dots) \quad (6.7)$$

become massless, the effective superpotential reads

$$\mathcal{W} = \sum_{i=1}^g \left\{ \sqrt{2} a_{Di} \tilde{M}_i M_i + \sum_{k=1}^{n_f} S_k^i m_k \tilde{M}_i M_i \right\} + \mu u_2(a_D, a) \quad (6.8)$$

where S_k^i is the k -th quark number charge of the i th dyon [2, 13]. Treating u_i as independent variables (or equivalently, a_{Di} 's), the equations of the minimum are

$$-\frac{\mu}{\sqrt{2}} = \sum_{i=1}^g \frac{\partial a_{Di}}{\partial u_2} \tilde{M}_i M_i; \quad 0 = \sum_{i=1}^g \frac{\partial a_{Di}}{\partial u_j} \tilde{M}_i M_i, \quad j = 3, 4, \dots, g+1; \quad (6.9)$$

$$\begin{aligned} \left(\sqrt{2} a_{D1} + \sum_{k=1}^{n_f} S_k^1 m_k \right) \tilde{M}_1 &= \left(\sqrt{2} a_{D1} + \sum_{k=1}^{n_f} S_k^1 m_k \right) M_1 = 0; \\ \dots &= \dots \\ \left(\sqrt{2} a_{Dg} + \sum_{k=1}^{n_f} S_k^g m_k \right) \tilde{M}_g &= \left(\sqrt{2} a_{Dg} + \sum_{k=1}^{n_f} S_k^g m_k \right) M_g = 0. \end{aligned} \quad (6.10)$$

The D -term constraint gives $|\tilde{M}_i| = |M_i|$. For generic hyperquark masses, Eqs.(6.9) require that

$$\tilde{M}_i, M_i \sim \sqrt{\mu \Lambda} \neq 0, \quad \forall i, \quad (6.11)$$

since $\frac{\partial a_{Di}}{\partial u_j}$ and $\frac{\partial a_{Dj}}{\partial u_i}$ obey no special relations. This means that

$$\sqrt{2} a_{Di} + \sum_{k=1}^{n_f} S_k^i m_k = 0 \quad (6.12)$$

for all i : i.e., all g monopoles are massless simultaneously. Condensation of each type of monopole, $\tilde{M}_i \neq 0$; $M_i \neq 0$, corresponding to the maximal Abelian subgroup of $SU(n_c)$ or of $USp(2n_c)$, amounts to confinement à la 't Hooft-Mandelstam-Nambu.

Actually, physical picture in the $m_i \rightarrow 0$ limit of these theories is more subtle and is far more interesting, as is discussed especially in Section 8.

It is in general difficult to determine explicitly the configurations $\{\phi_a\}$ which satisfy the $N = 1$ criterion mentioned above, although in some special cases (with $m_i = 0$) they can be found explicitly. We approach the problem by first setting $m_i = 0, \forall i$, and by perturbing the solutions for $\{\phi_a\}$ by considering the effects of m_i to first nontrivial orders.

It turns out that the $N = 1$ vacua of the $SU(n_c)$ and $USp(2n_c)$ gauge theories can all be generated from the various classes [8, 9, 17, 18] of superconformal theories with $m_i = \mu = 0$, by perturbing them with masses m_i (as well as with μ).

Some qualifying remarks are in order. In the case of $SU(n_c)$ theories we find that the first group of $N = 1$ vacua surviving the the adjoint mass perturbation ($\mu \text{Tr} \Phi^2$) and leading to finite VEVs,

arise from the class 1 ($r < n_f/2$), class 3 ($r = n_f/2$ with $n_c - n_f/2$ odd) and class 4 ($r = n_f/2$ with $n_c - n_f/2$ even) CFT, according to the classification of Eguchi et. al [9]. The second group of vacua, present only for $n_f > n_c$, on the other hand, arise from the so-called baryonic branch root [17] (see Eq.(6.18)): these latter CFT can also be regarded as special case of class 1 theories. In these vacua the flavor symmetry is unbroken in the $m_i \rightarrow 0$ limit.

It is important that the number of the vacua and qualitative results about symmetry breaking in each of them, can be obtained this way even when the unperturbed solution for $\{\phi_a\}$ is not known explicitly.

6.1 Superconformal Points and $N = 1$ Vacua for $SU(n_c)$

It will be seen that the first group of vacua (with multiplicity \mathcal{N}_1) are associated to the points,

$$\text{diag } \phi = (\underbrace{0, 0, \dots, 0}_r, \phi_{r+1}^{(0)}, \dots, \phi_{n_c}^{(0)}), \quad \sum_{a=r}^{n_c} \phi_a^{(0)} = 0, \quad (6.13)$$

in Eq.(6.1), with $\phi_a^{(0)}$'s chosen such that the nonzero $2(n_c - r - 1)$ branch points are paired¹³. The curves are of the form,

$$y^2 \sim x^{2r} (x - \beta_{01})^2 \cdots (x - \beta_{0, n_c - r - 1})^2 (x - \gamma_0)(x - \kappa_0), \quad r = 0, 1, 2, \dots, [n_f/2]. \quad (6.14)$$

For $r < n_f/2$ they correspond to the so-called class 1 superconformal theories [9].

The special cases with $r = n_f/2$, with $n_c - n_f/2$ odd, and $r = n_f/2$, $n_c - n_f/2$ even, may be interpreted as belonging to class 3 and class 4 [9], respectively. In fact, in these special cases the explicit configuration of ϕ_a 's can be found by using the method of [25]. The curve for bare quark masses with $r = n_f/2$ vanishing ϕ_a 's is:

$$y^2 = x^{n_f} \left[\prod_{k=1}^{n_c - n_f/2} (x - \phi_k)^2 - 4\Lambda^{2n_c - n_f} \right]. \quad (6.15)$$

We identify the first term in the square bracket with $(2\Lambda^{n_c - n_f/2} T_{n_c - n_f/2}(x/2\Lambda))^2$, where $T_N(x)$ is the Chebyshev polynomial of order N , $T_N(x) = \cos(N \cos^{-1} x)$, implying that

$$\phi_k = 2\Lambda \cos \pi(2k - 1)/2(n_c - n_f/2), \quad k = 1, \dots, n_c - n_f/2. \quad (6.16)$$

The two terms in the square bracket combine as

$$\begin{aligned} y^2 &= x^{n_f} \left[\left(2\Lambda^{n_c - n_f/2} T_{n_c - n_f/2} \left(\frac{x}{2\Lambda} \right) \right)^2 - 4\Lambda^{2n_c - n_f} \right] \\ &= -4x^{n_f} \Lambda^{2n_c - n_f} \sin^2 \left[\left(n_c - \frac{n_f}{2} \right) \arccos \frac{x}{2\Lambda} \right]. \end{aligned} \quad (6.17)$$

There are n_f -th order zero at $x = 0$, and double zeros at $x = 2\Lambda \cos \pi k / (n_c - n_f/2)$ for $k = 1, \dots, n_c - n_f/2 - 1$, and single zeros at $x = \pm 2\Lambda$. Note that for $r = n_f/2$, $n_c - n_f/2$ even, the zero at $x = 0$ is actually of order $n_f + 1$.

¹³Actually there is no vacuum of this type Eq.(6.14) with $r = \tilde{n}_c$. This will be shown in Appendix D.

Since these adjoint VEVs leading to Eq.(6.14) or Eq(6.17) break the discrete symmetry spontaneously, they appear in $2n_c - n_f$ copies.¹⁴ When (generic) quark masses are turned on, these vacua split into $n_f C_r$ -plet of single vacua. The vacua Eq.(6.14), Eq(6.17), correspond to what is called “nonbaryonic” branch roots in [17].

The second group of vacua will be related to the (trivial) superconformal theory

$$y^2 = x^{2\tilde{n}_c} (x^{n_c - \tilde{n}_c} - \Lambda^2)^2, \quad \tilde{n}_c = n_f - n_c, \quad (6.18)$$

corresponding to the adjoint configuration

$$\text{diag } \phi = (\underbrace{0, 0, \dots, 0}_{\tilde{n}_c}, \Lambda \omega^2, \dots, \Lambda \omega^{2(n_c - \tilde{n}_c)}) \quad (6.19)$$

with $\omega = e^{\pi i / (n_c - \tilde{n}_c)}$. This is the “baryonic” root of [17].

To justify these statements we must solve the following problem of purely mathematical nature. Suppose that a configuration $\{\phi_a\} = \{\phi_a^0\}$ has been found such that the curve of the $m_i = 0$ theory reduces to one of the forms, Eq.(6.14) or Eq.(6.18). Now we add small generic bare hyperquark masses m_i ; we want to determine the shifts of $\{\phi_a\}$, $\{\phi_a\} = \{\phi_a^0 + \delta\phi_a\}$, with constraints $\sum_a \delta\phi_a = 0$ and $\delta\phi_a \rightarrow 0$ as $m_i \rightarrow 0$, such that the massive curve Eq.(6.1) or Eq.(6.2) is maximally singular (with $g = n_c - 1$ double branch points). How many such solutions are there?

It turns out that this problem must be treated separately for several different cases: the result of this mass perturbation theory, which will be developed in the last section of this paper (Section 9), can be summarized as follows. For small number of the flavors ($n_f \leq n_c$) the total number of the $N = 1$ vacua generated by the mass perturbation from various conformal invariant points Eq.(6.14), Eq.(6.13), is:

$$\mathcal{N}_1 = (2n_c - n_f) \sum_{r=0}^{(n_f-1)/2} n_f C_r = (2n_c - n_f) 2^{n_f-1}, \quad (n_f = \text{odd}) \quad (6.20)$$

$$\mathcal{N}_1 = (2n_c - n_f) \sum_{r=0}^{n_f/2-1} n_f C_r + \frac{2n_c - n_f}{2} n_f C_{n_f/2} = (2n_c - n_f) 2^{n_f-1}, \quad (n_f = \text{even}), \quad (6.21)$$

which exhausts \mathcal{N} , Eq.(3.24). In Eq.(6.21) we have taken into account the fact that for even n_f , the vacua with $r = n_f/2$ do not transform under $Z_{2n_c - n_f}$ but only under $Z_{n_c - n_f/2}$. For larger n_f ($n_f \geq n_c + 1$), there are also

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c-1} (\tilde{n}_c - r) n_f C_r. \quad (6.22)$$

vacua coming from the “baryonic root”, Eq.(6.18), Eq.(6.19). The total $\mathcal{N}_1 + \mathcal{N}_2$ correctly matches the known total number of the vacua. The arithmetics is the same as in Eq.(4.49), Eq.(4.50), and will not be repeated.

¹⁴There is an exception to this. In the case of $r = n_f/2$ with n_f even, the explicit configuration of ϕ_a ’s (Section 9.1) shows that the vacuum respects Z_2 subgroup of the $Z_{2n_c - n_f}$ symmetry, showing that it appears in $n_c - n_f/2$ copies rather than $2n_c - n_f$. This fact is crucial in the vacuum counting below Eq.(6.21).

Actually, there is an interesting subtlety in this vacuum counting. In the first group of vacua Eq.(6.20), Eq.(6.21), the term $r = \tilde{n}_c = n_f - n_c$ must be dropped. The vacuum Eq.(6.14) turns out to be nonexistent for $r = \tilde{n}_c = n_f - n_c$, see Appendix D. At the same time, however, the mass perturbation around the baryonic branch root Eq.(6.18), Eq.(6.19), gives (see Sections 8, 9)

$$\mathcal{N}_2 + (n_c - \tilde{n}_c) n_f C_{\tilde{n}_c} \quad (6.23)$$

vacua, the second term of which compensates precisely the missing term in the sum in \mathcal{N}_1 .

6.2 Superconformal Points and $N = 1$ Vacua of $USp(2n_c)$ Theory

The general curve of $USp(2n_c)$ theories is given by

$$xy^2 = \left[x \prod_{a=1}^{n_c} (x - \phi_a^2) + 2\Lambda^{2n_c+2-n_f} m_1 \cdots m_{n_f} \right]^2 - 4\Lambda^{2(2n_c+2-n_f)} \prod_{i=1}^{n_f} (x + m_i^2). \quad (6.24)$$

When $2n_c + 2 - n_f = 0$, the theory is superconformal and $2\Lambda^{2n_c+2-n_f}$ in the above expression is replaced by

$$g(\tau) = \frac{\vartheta_2^4}{\vartheta_3^4 + \vartheta_4^4}. \quad (6.25)$$

It turns out that the first group of $N = 1$ vacua can be generated from the CFT described by Chebyshev solutions for $m_i = 0$ theory.

When n_f is odd, choose $(n_f - 1)/2$ eigenvalues $\phi_1 = \cdots = \phi_{(n_f-1)/2}$ vanishing. Then the curve becomes

$$y^2 = x^{n_f-1} \left[x \prod_{k=1}^{n_c-(n_f-1)/2} (x - \phi_k^2)^2 - 4\Lambda^{2(2n_c+2-n_f)} \right]. \quad (6.26)$$

We identify the first term in the square bracket with $(2\Lambda^{2n_c+2-n_f} T_{2n_c+2-n_f}(\sqrt{x}/2\Lambda))^2$, where $\phi_k^2 = 4\Lambda^2 \cos^2 \pi(2k-1)/2(2n_c+2-n_f)$ for $k = 1, \dots, n_c - (n_f - 1)/2$. Then the two terms in the square bracket combine as

$$\begin{aligned} y^2 &= x^{n_f-1} \Lambda^{2(2n_c+2-n_f)} \left[4T_{2n_c+2-n_f}^2 \left(\frac{\sqrt{x}}{2\Lambda} \right) - 4 \right] \\ &= -4x^{n_f-1} \Lambda^{2(2n_c+2-n_f)} \sin^2 \left[(2n_c + 2 - n_f) \arccos \frac{\sqrt{x}}{2\Lambda} \right]. \end{aligned} \quad (6.27)$$

There are $n_f - 1$ zeros at $x = 0$, and double zeros at $x = 4\Lambda^2 \cos \pi k / (2n_c + 2 - n_f)$ for $k = 1, \dots, n_c - (n_f - 1)/2$, and a single zero at $x = 4\Lambda^2$.

When n_f is even, choose $n_f/2 - 1$ eigenvalues $\phi_1 = \cdots = \phi_{n_f/2-1}$ vanishing. Then the curve becomes

$$y^2 = x^{n_f-1} \left[\prod_{k=1}^{n_c+1-n_f/2} (x - \phi_k^2)^2 - 4\Lambda^{2(2n_c+2-n_f)} \right]. \quad (6.28)$$

We identify the first term in the square bracket with $(2\Lambda^{2n_c+2-n_f}T_{2n_c+2-n_f}(\sqrt{x}/2\Lambda))^2$, where $\phi_a^2 = 4\Lambda^2 \cos^2 \pi(2k-1)/2(2n_c+2-n_f)$ for $k = 1, \dots, n_c+1-n_f/2$. Then the two terms in the square bracket combine as

$$\begin{aligned} y^2 &= x^{n_f-1} \Lambda^{2(2n_c+2-n_f)} \left[4T_{2n_c+2-n_f}^2 \left(\frac{\sqrt{x}}{2\Lambda} \right) - 4 \right] \\ &= -4x^{n_f-1} \Lambda^{2(2n_c+2-n_f)} \sin^2 \left[(2n_c+2-n_f) \arccos \frac{\sqrt{x}}{2\Lambda} \right]. \end{aligned} \quad (6.29)$$

Since the \sin^2 factor gives a single zero at $x = 0$, there are n_f zeros at $x = 0$, and double zeros at $x = 4\Lambda^2 \cos \pi k / (2n_c+2-n_f)$ for $k = 1, \dots, n_c - n_f/2$, and a single zero at $x = 4\Lambda^2$.

In the absence of quark masses, the theory is invariant under $Z_{2n_c+2-n_f}$ symmetry: $x \rightarrow e^{2\pi i/(2n_c+2-n_f)}x$, $\phi_a^2 \rightarrow e^{2\pi i/(2n_c+2-n_f)}\phi_a^2$. Therefore the Chebyshev solutions discussed here appear $2n_c+2-n_f$ times.

$USp(2n_c)$ theories also have special Higgs branch roots similar to the baryonic roots of the $SU(n_c)$ theories. This baryonic-like root is obtained in the $m_i = 0$ limit, by setting $\phi_1, \dots, \phi_{n_c-\tilde{n}_c} \neq 0$, $\phi_{n_c-\tilde{n}_c+1} = \dots = \phi_{n_c} = 0$. Here and below, $\tilde{n}_c = n_f - n_c - 2$. The curve Eq. (6.24) becomes

$$y^2 = x^{2\tilde{n}_c+1} \prod_{k=1}^{n_c-\tilde{n}_c} (x - \Phi_k)^2 - 4\Lambda^{4n_c+4-2n_f} x^{n_f-1}. \quad (6.30)$$

We take

$$(\Phi_1^2, \dots, \Phi_k^2, \dots, \Phi_{n_c-\tilde{n}_c}^2) = \Lambda^2(\omega, \dots, \omega^{2k-1}, \dots, \omega^{2(n_c-\tilde{n}_c)-1}), \quad (6.31)$$

where $\omega = e^{\pi i/(n_c-\tilde{n}_c)}$. Note that our ω is the square root of ω in [17] because of later convenience. Then the product $\prod_{k=1}^{n_c-\tilde{n}_c} (x - \Phi_k)$ can be rewritten as $x^{n_c-\tilde{n}_c} + \Lambda^{2(n_c-\tilde{n}_c)}$, and the curve becomes

$$\begin{aligned} y^2 &= x^{2\tilde{n}_c+1} \left[(x^{n_c-\tilde{n}_c} + \Lambda^{2(n_c-\tilde{n}_c)})^2 - 4\Lambda^{4n_c+4-2n_f} x^{n_c-\tilde{n}_c} \right] \\ &= x^{2\tilde{n}_c+1} (x^{n_c-\tilde{n}_c} - \Lambda^{2(n_c-\tilde{n}_c)})^2. \end{aligned} \quad (6.32)$$

The double zeros of the factor in the parenthesis are at

$$x = \Lambda^2 \omega^2, \Lambda^2 \omega^4, \dots, \Lambda^2 \omega^{2k}, \dots, \Lambda^2 \omega^{2(n_c-\tilde{n}_c)} = \Lambda^2. \quad (6.33)$$

When the quark masses are turned on, these points split. We require again that the shift of ϕ 's be such that the full curve remains maximally singular (with maximal possible number of double roots). The result of mass perturbation analysis, given in Section 9, can be summarized as follows. There are two groups of $N = 1$ vacua predicted by the Seiberg-Witten curve in $USp(2n_c)$ theories. The Chebyshev point Eq.(6.27), Eq.(6.28), spawns $\mathcal{N}_1 = (2n_c+2-n_f) \cdot 2^{n_f-1}$ vacua upon mass perturbation, while the special point Eq.(6.30), Eq.(6.31), splits into $\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c} (\tilde{n}_c - r + 1) n_f C_r$ vacua. Their sum coincides with the total number of $N = 1$ vacua found from the semiclassical as well as from large μ analyses.

7 Numerical Study of $N = 1$ Vacua in $SU(3)$ and $USp(4)$ Theories

As a way of checking these results and of illustrating some of their features, we have performed a study of the rank 2 theories, by numerically determining the points in QMS where the curves $y^2 = G(x)$ of Eq.(6.1)-Eq.(6.3) become maximally singular. This has been done by solving the equation

$$\mathcal{R}\left(G(x), \frac{dG(x)}{dx}\right) = 0, \quad (7.1)$$

where \mathcal{R} stands for the resultant, using Mathematica.

7.1 $SU(3)$ theory with $n_f = 1 \sim 5$

i) In the case with $n_f = 1$, one expects from Eq. (3.23) $\mathcal{N} = 5$. From the known curve,

$$y^2 = (x^3 - ux - v)^2 - \Lambda_1^5 (x + m), \quad (7.2)$$

one finds indeed five vacua, related by an approximate Z_5 symmetry. See Fig. 1 for $\Lambda_1 = 2$ and $m = 1/64$.

ii) In the case with $n_f = 2$ one expects eight vacua, related by an approximate Z_4 symmetry which transform u and v as $u \rightarrow -u$ and $v \rightarrow \exp(\frac{3}{2}i\pi)v$. One finds indeed eight singularities, grouped into two approximate doublets, and four singlets for generic small masses. (see Fig. 2).

iii) For $n_f = 3$, one gets $\mathcal{N} = 12$, with Z_3 symmetry. The numerical analysis with the curve:

$$y^2 = (x^3 - ux - v)^2 - \Lambda_3^3 (x + m_1)(x + m_2)(x + m_3). \quad (7.3)$$

shows that there are indeed twelve vacua satisfying the criterion of the two mutually local dyons becoming massless, and they are found in roughly three groups of triplets and three singlets of singularities. In the equal mass limit each of the three triplets coalesce to a point in the (u, v) space, showing that the massless monopoles there are in the representation $\underline{3}$ of $SU(3)$, while those at the other vacua are singlets (see Fig. 3), in complete agreement with the analysis of previous sections.

iv) For $n_f = 4$, Eq. (3.23) gives $\mathcal{N} = 17$ vacua. It is reassuring that one indeed finds from Eq.(7.1) seventeen vacua for generic and unequal masses¹⁵. At small masses these vacua are grouped into an approximate sextet, two quartets and three singlets, suggesting the assignments of rank 2, 1 and 0 antisymmetric representation of $SU(4)$ global flavor group. The number of the vacua (six) in the limit of equal masses is consistent with this assignment (see Fig. 4)

In a quartet vacuum, the condensation of the monopoles breaks the $SU(4)$ symmetry to $U(3)$, while in a singlet vacua the flavor symmetry remains unbroken.

¹⁵As a further check, we verified the number of the vacua by using another parametrization of the curve given by Minahan et al. [26].

Something very interesting happens in the sextet vacua. Namely, in the equal mass or massless limit, we find that four branch points in the x plane coalesce, suggesting conformal invariant vacuum. In fact, this result was to be expected, since these “sextet” vacua are examples of the particular class (“class 3”) of nontrivial conformal invariant theories studied in [9], where $2r$ branch points in the x plane coalesce, where $r = n_f/2$. This is known to occur for $SU(3)$ theories with $n_f = 4$ at special values of u and v when the quark masses are all equal (m): their precise position is [9] $U = 3M^2$, and $V = 2M^3$, where $u = (7 + 24M + 36M^2)/12$ and $v = (11 + 45M + 54M^2 + 54M^3)/27$ and $M = m - 1/3$, and the curve becomes

$$y^2 = (x + m)^4(x + 1 - 2m)(x - 1 - 2m). \quad (7.4)$$

The values of u, v found by us numerically from the criterion of $N = 1$ vacua precisely match these values, showing that this particular conformal vacuum survives the $N = 1$ perturbation.

In order to determine which particles are actually present, it is necessary to study the monodromy transformation properties of $(a_{D1}, a_{D2}, a_1, a_2)$ [1, 2, 3] around this particular singularity. In Appendix E we present such an analysis (the analysis is actually done for all seventeen vacua of $SU(3)$, $n_f = 4$ model). Our result shows that the massless particles (in the $\mu = 0$ limit) present at this singularity have quantum numbers

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (0, 1, 0, 1), (1, 2, 2, 0), (1, 1, 0, 1). \quad (7.5)$$

As these particles are relatively nonlocal, such a vacuum is conformal invariant [8].

This is an example of a very general phenomenon, already discussed in the previous section.

- v) Finally, for $n_f = 5$ we verified (for large masses) the presence of twenty-three quantum vacua, in accordance with Eq. (3.23). From the curve:

$$y^2 = \prod_{a=1}^{n_c} (x - \phi_a)^2 - 4\Lambda_5 \prod_{i=1}^5 \left(x + m_i + \frac{\Lambda_5}{n_c} \right) = (x^3 - ux - v)^2 - 4\Lambda_5 \prod_{i=1}^5 \left(x + m_i + \frac{\Lambda_5}{n_c} \right), \quad (7.6)$$

in the small mass limit, the grouping of the singularities found is compatible with the assignment into a decuplet, two quintets and three singlets of (approximate) global $SO(5)$ symmetry. The number of vacua (six) in the equal mass case, is in agreement with this structure.

In particular, the decuplet vacua corresponds to the curve,

$$y^2 = \left(x + m + \frac{\Lambda_5}{n_c} \right)^4 \left[\left(x - \frac{2\Lambda_5}{n_c} - 2m \right)^2 - 4\Lambda_5 \left(x + m + \frac{\Lambda_5}{n_c} \right) \right], \quad (7.7)$$

and corresponds to the class 1 (trivial) conformal field theory [9].

Furthermore, since $n_f > n_c + 1$ in this case, the theory belongs to the “large n_f ” class of Sec. 4.3 we expect $\mathcal{N}_2 = 7$ (see Eq. (4.47)) of vacua to show particular properties. We find that indeed seven of the vacua (in the equal mass limit, a quintet and two singlets) approach the form

$$y^2 = x^4(x - \Lambda)^2 \quad (7.8)$$

in the $m_i \rightarrow 0$ limit, after a shift in x . This is exactly the structure of the singularities at the root of the baryonic branch[17].

7.2 $USp(2n_c)$ with $2n_f$ Flavors

- i) For $USp(4)$ with $n_f = 1$, we do find five vacua, as shown in Fig. 5, consistently with the approximate Z_5 symmetry.
- ii) For $n_f = 2$ with the same gauge group, eight quantum vacua are found to group into four doublets, showing that monopoles appear in the fundamental representation $(2, 1)$ and $(1, 2)$, of the flavor group $SO(2n_f) \sim SU(2) \times SU(2)$. Their condensation leaves a $U(2)$ subgroup invariant, in accordance with the naïve expectation. In the equal mass limit one of the $SU(2)$ symmetries is exact. It is seen that in this case the $SU(2)$ symmetry is broken spontaneously in four of the vacua, while in four others it is not, in perfect agreement with what was found in Sec. 4 (see Fig. 6).
- iii) For the $USp(4)$ theory with $n_f = 3$, the formula (3.44) and the discrete symmetry suggests that monopoles form a quartet (12 vacua grouped into three quartets of nearby singularities). This is indeed the case as shown in Fig. 7. The condensation of $\underline{4}$ of $SO(6) \sim SU(4)$ breaks the chiral symmetry to $U(3)$, as expected.
- iv) The theory with four flavors (and the same gauge group) has seventeen vacua. In the massless limit, they group in two spinors (octets) and one singlet of the global symmetry $SO(8)$. In the case of degenerate but nonvanishing masses, the spinors $\mathbf{8}$ of $SO(8)$ can split in two possible ways: $\mathbf{1} + \mathbf{6} + \mathbf{1}$ of $U(4)$ in one case, and with $\mathbf{4} + \mathbf{4}^*$ of $U(4)$ in the other. This is indeed the situation shown in Fig. 8. This shows the correctness of our assignment and that in the massless limit the condensation of the monopole in the spinor representation of $SO(8)$ breaks the chiral symmetry to $U(4)$.
- v) Finally, we verified that the theory $USp(4)$ with five flavors has indeed twenty-three quantum vacua.

8 Effective Lagrangian Description of $N = 1$ Vacua at Small μ

A deeper insight into physics at small m_i and μ can be obtained by re-examining the works of Argyres, Plesser and Seiberg [17] and Argyres, Plesser and Shapere [18], who showed how the non-renormalization theorem of the hyperKähler metric on the Higgs branch could be used to show the persistence of unbroken non-abelian gauge group at the “roots” of the Higgs branches where they intersect the Coulomb branch. In fact, they found two kinds of such submanifolds, called “non-baryonic branch” (or mixed-branch) roots, and “baryonic branch” roots (these terminologies refer specifically to the $SU(n_c)$ theory, but the situation is similar also in $USp(2n_c)$ theory). The latter is present only for larger values of the flavor ($n_f > n_c$) while the former exists always.

Below, we show how the low-energy effective action description of [17, 18] match our findings of Sections 3 - 7, after correcting a few errors and clarifying some issues left unclear there. In doing so, a very clear and interesting picture of the infrared dynamics of our theories emerges, which was summarized in the Introduction.

Let us discuss the $SU(n_c)$ theories first.

8.1 $SU(n_c)$

The non-baryonic roots are further classified into sub-branches characterized by an unbroken $SU(r) \times U(1)^{n_c-r}$ gauge symmetry for $r \leq [n_f/2]$, with n_f flavor of massless hypermultiplets in the fundamental representation of $SU(r)$, as well as $n_c - r - 1$ singlet “monopole” hypermultiplets having charges only in the $U(1)^{n_c-r}$ gauge sector. Their quantum numbers are shown in Table 4 taken from [17].

Upon turning on the $\mu\Phi^2$ perturbation, the effective superpotential of the theory is, according to Argyres, Plesser and Seiberg [17],

$$W_{non\ bar} = \sqrt{2}\text{Tr}(q\phi\tilde{q}) + \sqrt{2}\psi_0\text{Tr}(q\tilde{q}) + \sqrt{2} \sum_{k=1}^{n_c-r-1} \psi_k e_k \tilde{e}_k + \mu \left(\Lambda \sum_{i=0}^{n_c-r-1} x_i \psi_i + \frac{1}{2}\text{Tr}\phi^2 \right), \quad (8.1)$$

where the last term arises from $\mu\Phi^2$ perturbation, ϕ referring to the $SU(r)$ part of the adjoint field and ψ_i being the $N = 2$ partner of the dual $U(1)_i$ gauge field; x_i are some constants. By minimizing

	$SU(r)$	$U(1)_0$	$U(1)_1$	\dots	$U(1)_{n_c-r-1}$	$U(1)_B$
$n_f \times q$	$\underline{\mathbf{r}}$	1	0	\dots	0	0
e_1	$\underline{\mathbf{1}}$	0	1	\dots	0	0
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
e_{n_c-r-1}	$\underline{\mathbf{1}}$	0	0	\dots	1	0

Table 4: The effective degrees of freedom and their quantum numbers at the “nonbaryonic root”.

the potential, one finds that supersymmetric $N = 1$ vacua exist for any r . (Actually, one assumed here that the maximum number of massless monopole-like particles e_k exist; such vacua are called in [17] “special points” of the non baryonic roots. Only these vacua survive the $N = 1$ perturbation. Some flat directions remain, as expected.)

We now perturb these theories at the nonbaryonic branch roots further with small hypermultiplet (quark) masses, m_i . Add the mass terms

$$\Delta W_{non\ bar} = m_i q_i \tilde{q}_i + \sum_{k=1}^{n_c-r-1} S_k^j m_j e_k \tilde{e}_k, \quad (8.2)$$

where S_k^j represents the j -th quark number of the “monopole” e_k [2, 13]. The part involving e_k , \tilde{e}_k and ψ_k , $k = 1, 2, \dots, n_c - r - 1$ is trivial and gives

$$\psi_k \sim \{m_i\}; \quad e_k = \tilde{e}_k \sim \sqrt{\mu\Lambda}. \quad (8.3)$$

The vacuum equations for other components are

$$0 = [\phi, \phi^\dagger]; \quad (8.4)$$

$$\nu \delta_a^b = q_a^i (q^\dagger)_i^b - (\tilde{q}^\dagger)_a^i \tilde{q}_i^b; \quad (8.5)$$

$$0 = q_a^i (q^\dagger)_i^a - (\tilde{q}^\dagger)_a^i \tilde{q}_i^a; \quad (8.6)$$

$$q_a^i \tilde{q}_i^b - \frac{1}{r} \delta_a^b (q_c^i \tilde{q}_i^c) + \sqrt{2} \mu \phi_a^b = 0; \quad (8.7)$$

$$0 = \sqrt{2} \phi_a^b q_b^i + q_a^i (m_i + \sqrt{2} \psi_0); \quad (\text{no sum over } i, a) \quad (8.8)$$

$$0 = \sqrt{2} \tilde{q}_i^b q_b^a + (m_i + \sqrt{2} \psi_0) \tilde{q}_i^b \quad (\text{no sum over } i, a). \quad (8.9)$$

$$\sqrt{2} \text{Tr}(q\tilde{q}) + \mu\Lambda = 0. \quad (8.10)$$

First diagonalize the Higgs scalar by color rotations.

$$\text{diag } \phi = (\phi_1, \phi_2, \dots, \phi_r), \quad \sum \phi_a = 0. \quad (8.11)$$

The equations Eq.(8.4)-Eq.(8.9) are formally identical to Eq.(3.5)-Eq.(3.9), with the replacements,

$$Q \rightarrow q; \quad , m_i \rightarrow m_i + \sqrt{2} \psi_0; \quad n_c \rightarrow r, \quad (8.12)$$

therefore we immediately find solutions classified by an integer ℓ ($\ell = 0, 1, \dots, r - 1$) where

$$\phi = \frac{1}{\sqrt{2}} \text{diag}(-m_1 - \sqrt{2} \psi_0, -m_2 - \sqrt{2} \psi_0, \dots, -m_\ell - \sqrt{2} \psi_0, c, c, \dots, c), \quad \sum \phi_a = 0; \quad (8.13)$$

$$q_a^i = \begin{pmatrix} d_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ d_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \vdots \\ d_\ell \\ 0 \\ \vdots \end{pmatrix}; \quad q_a^i = 0, \quad i = \ell + 1, \dots, n_f. \quad (8.14)$$

$$\tilde{q}_i^a = \begin{pmatrix} \tilde{d}_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{d}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ \tilde{d}_\ell \\ 0 \\ \vdots \end{pmatrix}; \quad \tilde{q}_i^a = 0, \quad i = \ell + 1, \dots, n_f, \quad (8.15)$$

where

$$c = \frac{1}{r - \ell} \sum_{k=1}^{\ell} (m_k + \sqrt{2}\psi_0) \quad (8.16)$$

and d_i, \tilde{d}_i 's are of order of $O(\sqrt{\mu(\psi_0 + m_i)})$. Note however that Eq.(8.10) is new: it fixes ψ_0 such that

$$\frac{\sqrt{2}r\mu}{r - \ell} \sum_{j=1}^{\ell} (\psi_0 + m_j) = -\mu\Lambda. \quad (8.17)$$

These are the first group of $N = 1$ solutions found in [17]. The fact that in the limit $m_i \rightarrow 0$,

$$\psi_0 \sim \Lambda, \quad (8.18)$$

however, shows that *these solutions, involving fluctuations much larger than both m_i and μ , lie beyond the validity of the low-energy effective Lagrangian*. They are therefore an artifact of the approximation, and must be discarded.

The correct $N = 1$ vacua are found instead by choosing $\ell = r$ in Eq.(8.13) and by selecting VEVS,

$$\psi_0 = -\frac{1}{\sqrt{2}r} \sum_i m_i, \quad (8.19)$$

$$d_i \tilde{d}_i = \mu \left(m_i - \frac{1}{r} \sum_j^r m_j \right) - \frac{\mu\Lambda}{\sqrt{2}r}; \quad (8.20)$$

$$e_k \tilde{e}_k = -\mu\Lambda, \quad (8.21)$$

with no c 's in Eq.(8.13) ¹⁶. These satisfy clearly all of the vacuum equations. These vacua are smoothly related to the unperturbed ones and therefore are reliable. In the massless limit this gives

$$d_i \sim \tilde{d}_i \sim \sqrt{\mu\Lambda}. \quad (8.22)$$

We find thus ${}_f C_r \cdot (2n_c - n_f)$ vacua with the correct symmetry breaking pattern,

$$U(n_f) \rightarrow U(r) \times U(n_f - r), \quad (8.23)$$

which is exactly what is expected from the analysis made at large μ . The multiplicity ${}_f C_r$ arises from the choice of r (out of n_f) quark masses used to construct the solution.

For $r = n_f/2$, the theory at the singularity becomes a non-trivial superconformal theory. There is no description of this singularity in terms of weakly coupled local field theory. The monodromy

¹⁶ Note that an analogous solution was not possible for Eq.(3.5)-Eq.(3.9), since the quark masses are generic and the adjoint field must be traceless.

around the singularity shows that the theory is indeed superconformal (we checked this explicitly for $n_c = 3$ and $n_f = 4$). Careful perturbation of the curve by the quark masses made in Section 9 shows that there are $(n_c - n_f/2)_{n_f} C_{n_f/2}$ vacua.

The total number of $N = 1$ vacua is then given by \mathcal{N}_1 of Eq.(6.20), Eq.(6.21), after summing over r . Actually, one must subtract the term $r = \tilde{n}_c$, since the nonbaryonic root with $r = \tilde{n}_c$ does not exist (see Appendix D).

The baryonic root is an $SU(\tilde{n}_c) \times U(1)^{n_c - \tilde{n}_c}$ theory ($\tilde{n}_c \equiv n_f - n_c$), with n_f massless quarks q and $n_c - \tilde{n}_c = 2n_c - n_f$ massless singlets. Their charges are summarized in Tab. 5. The effective

	$SU(\tilde{n}_c)$	$U(1)_1$	\dots	$U(1)_{n_c - \tilde{n}_c}$	$U(1)_B$
$n_f \times q$	$\underline{\tilde{n}_c}$	$1/\tilde{n}_c$	\dots	$1/\tilde{n}_c$	$-n_c/\tilde{n}_c$
e_1	$\underline{1}$	-1	\dots	0	0
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
$e_{n_c - \tilde{n}_c}$	$\underline{1}$	0	\dots	-1	0

Table 5: The effective degrees of freedom and their quantum numbers at the “baryonic root”, taken from [17].

action in this case, upon the $N = 1$ perturbation, is

$$W_{bar} = \sqrt{2} \text{Tr}(q\phi\tilde{q}) + \frac{\sqrt{2}}{\tilde{n}_c} \text{Tr}(q\tilde{q}) \left(\sum_{k=1}^{n_c - \tilde{n}_c} \psi_k \right) - \sqrt{2} \sum_{k=1}^{n_c - \tilde{n}_c} \psi_k e_k \tilde{e}_k + \mu \left(\Lambda \sum_{i=1}^{n_c - \tilde{n}_c} x_i \psi_i + \frac{1}{2} \text{Tr} \phi^2 \right). \quad (8.24)$$

Again, we add the quark mass terms

$$\text{Tr}(mq) - \sum_{k,i} S_k^i m_i e_k \tilde{e}_k \quad (8.25)$$

and minimize the potential. The equations are:

$$q_a^i q_i^{\dagger b} - \tilde{q}_a^{\dagger i} \tilde{q}_i^b = \nu \delta_a^b; \quad (8.26)$$

$$q_a^i \tilde{q}_i^b - \frac{1}{\tilde{n}_c} \delta_a^b (q_c^i \tilde{q}_i^c) + \mu \phi_a^b = 0; \quad (8.27)$$

$$q_a^i m_i + \sqrt{2} \phi_a^b q_b^i + \frac{\sqrt{2}}{\tilde{n}_c} q_a^i \sum_k \psi_k = 0; \quad (8.28)$$

$$\tilde{q}_i^a m_i + \sqrt{2} \phi_b^a \tilde{q}_i^b + \frac{\sqrt{2}}{\tilde{n}_c} \tilde{q}_i^a \sum_k \psi_k = 0; \quad (8.29)$$

$$\frac{\sqrt{2}}{\tilde{n}_c} \text{Tr}(\tilde{q}q) - \sqrt{2} e_k \tilde{e}_k + \mu \Lambda x_k = 0; \quad (8.30)$$

$$(\psi_k - S_k^i m_i) e_k = 0; \quad (\psi_k - S_k^i m_i) \tilde{e}_k = 0, \quad (8.31)$$

and

$$\phi = \frac{1}{\sqrt{2}} \text{diag}(-m_1, \dots, -m_r, c, \dots, c); \quad c = \frac{1}{\tilde{n}_c - r} \sum_{k=1}^r m_k. \quad (8.32)$$

We find two types of vacua. The first type has $e_k = \tilde{e}_k = (\mu \Lambda x_k / \sqrt{2})^{1/2}$ for all $k = 1, \dots, n_c - \tilde{n}_c$. Minimizing the potential in this case, we find

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c-1} (\tilde{n}_c - r) {}_{n_f}C_r \quad (8.33)$$

$N = 1$ vacua, characterized by the VEVs Eq.(8.32) and

$$d_i, \tilde{d}_i \sim \sqrt{\mu m} \xrightarrow{m_i \rightarrow 0} 0, \quad e_k, \tilde{e}_k \sim \sqrt{\mu \Lambda}. \quad (8.34)$$

The unbroken $SU(\tilde{n}_c - r)$ gauge group gives $\tilde{n}_c - r$ vacua each. These vacua describe the vacua with unbroken $U(n_f)$ symmetry, which are known to exist from the large μ analysis. The total number of vacua of this group found here agrees with Eq.(6.22) at the large μ regime.

The second type of vacua in Eqs. (8.24,8.25) has one of the $e_k = \tilde{e}_k = 0$ (hence $n_c - \tilde{n}_c = 2n_c - n_f$ choices) while $\partial W / \partial \psi_k = 0$ requires quarks to condense with $q = \tilde{q} \sim \sqrt{\mu \Lambda}$. Dropping $e_k = \tilde{e}_k = 0$ from the Lagrangian, it becomes the same as that of the non-baryonic root Eqs. (8.1,8.2) and gives $(2n_c - n_f) {}_{n_f}C_{\tilde{n}_c}$ vacua. This precisely compensates the exclusion of $r = \tilde{n}_c$ in the sum for the non-baryonic roots and the correct total number of vacua $\mathcal{N}_1 + \mathcal{N}_2$ is obtained.

We thus find that both the number and the symmetry properties of the $N = 1$ theories at small adjoint mass μ match exactly those found at large μ . For vacua with $r = 1$, the “quarks” in the effective Lagrangian (8.1) are nothing but the $U(1)^{n_c-1}$ monopoles in the fundamental representation of $U(n_f)$; this is checked by studying the monodromy around the singularity which showed that the “quarks” are indeed magnetically charged. Therefore the standard picture of confinement and flavor symmetry breaking by condensation of flavor-non-singlet monopoles is valid for these vacua.

For general vacua $r > 1$ associated with various nonbaryonic roots, the effective Lagrangian (8.1) describes correctly the physics of $N = 1$ vacua at small μ , in terms of *magnetic quarks* of a non Abelian $SU(r) \times U(1)^{n_c-r}$ theory. In contrast to the $r = 1$ case, these quarks cannot be identified with the semiclassical monopoles of the maximally Abelian $U(1)^{n_c-1}$ group. Note that the condensation of such monopoles in the rank- r anti-symmetric tensor representation, which might be suggested from the number of the singularities which group into a nearby cluster in the small m_i limit and at the same time from the semiclassical analysis (see Appendix B), would have yielded the correct pattern of symmetry breaking; at the same time, however, it would have led to an uncomfortably large number of Nambu-Goldstone bosons associated to the accidental $SU(n_f C_r)$ symmetry. The system avoids this paradox elegantly, by having magnetic quarks as low-energy degrees of freedom and having these condensed. These facts, and the comparison of their quantum numbers, lead us to conclude that, as we approach the non-baryonic roots from semi-classical (large VEV) region on the Coulomb branch, the semi-classical monopoles in the rank- r anti-symmetric tensor representation are smoothly matched to “baryons” of the $SU(r)$ theory,

$$\epsilon^{a_1 \dots a_r} q_{a_1}^{i_1} q_{a_2}^{i_2} \dots q_{a_r}^{i_r}, \quad (8.35)$$

and break up (the system being infrared-free) into weakly coupled magnetic quarks, before becoming massless.

The case $r = n_f/2$ is exceptional and highly non-trivial. Although the analysis leading to Eq.(8.19), Eq.(8.22), is formally valid in this case also, physics is really different. The low energy degrees of freedom involve relatively nonlocal states, arising from the nontrivial, class 3 CFT [9]. In this case, the theory is in the same universality class as the finite $SU(n_f/2)$ theories. For an explicit check with monodromy, see Appendix E, for the simplest example of this type, $r = 2$ vacua of the $SU(3)$ theory with $n_f = 4$.

As for the second group of vacua with vanishing VEVs found at the “baryonic root”, they are in the free-magnetic phase, with observable (magnetic) quarks, weakly interacting with $SU(n_f - n_c)$ gauge fields.

8.2 $USp(2n_c)$

In the $USp(2n_c)$ gauge theories, the first type of vacua can be identified more easily by first considering the equal but nonvanishing quark masses. The adjoint VEVs in the curve Eq.(6.24) can be chosen so as to factor out the behavior

$$y^2 = (x + m^2)^{2r} [\dots], \quad r = 1, 2, \dots \quad (8.36)$$

which describes an $SU(r) \times U(1)^{n_c - r + 1}$ gauge theory with n_f quarks. See Section 9.2. These (trivial) superconformal theories belong in fact to the same universality classes as those found in the $SU(n_c)$ gauge theory, as pointed out by Eguchi and others [9]. They are therefore described by exactly the same Lagrangian Eq.(8.1). At each vacuum with r , the symmetry (of equal mass theory, $U(n_f)$) is broken spontaneously as

$$U(n_f) \rightarrow U(r) \times U(n_f - r). \quad (8.37)$$

When a small mass splitting is added among m_i ’s, each of the r vacuum splits into $n_f C_r$ vacua, leading to the total of

$$(2n_c + 2 - n_f) \sum_{r=0}^{(n_f-1)/2} n_f C_r = (2n_c + 2 - n_f) 2^{n_f-1}, \quad (n_f = \text{odd}) \quad (8.38)$$

$$(2n_c + 2 - n_f) \sum_{r=0}^{n_f/2-1} n_f C_r + \frac{2n_c + 2 - n_f}{2} n_f C_{n_f/2} = (2n_c + 2 - n_f) 2^{n_f-1}, \quad (n_f = \text{even}), \quad (8.39)$$

vacua of this type, consistently with Eq. (3.44). These $N = 1$ vacua seem to have been overlooked in [18] altogether¹⁷.

In the massless limit the underlying theories possess a larger, flavor $SO(2n_f)$ symmetry. We know also from the large μ analysis that in the first group of vacua (with finite vevs), this symmetry

¹⁷With a hindsight, we see that this was inevitable: as we show below, there is no local effective Lagrangian description of low-energy physics in the $m_i = 0$ limit and Argyres, Plesser and Shapere [18] worked precisely in such a regime.

is broken spontaneously to $U(n_f)$ symmetry always. How can such a result be consistent with Eq.(8.37) of equal (but nonvanishing) mass theory?

What happens is that in the massless limit various $N = 1$ vacua with different symmetry properties Eq.(8.37) (plus eventually other singularities) coalesce. The location of this singularity can be obtained exactly in terms of Chebyshev polynomials, and is given by Eqs. (6.26)-(6.29). At the singularity there are mutually non-local dyons and hence the theory is at a non-trivial infrared fixed point. In the example of $USp(4)$ theory with $n_f = 4$, we have explicitly verified this by determining the singularities and branch points at finite equal mass m and by studying the limit $m \rightarrow 0$. There is no description in terms of a weakly coupled local field theory, just as in the case $r = n_f/2$ for $SU(n_c)$ theories. Since the global flavor symmetry is $SO(2n_f)$, *these superconformal theories belong to a different universality class as compared to those at finite mass or to those of $SU(n_c)$ theories.* When n_f is even, the theory is in the same universality class as the finite $USp(n_f - 2)$ theories. When n_f is odd, the theory is in a different universality class of “strong-coupled conformal theories.”

We find this behavior reasonable because the semi-classical monopoles are in the spinor representation of the $SO(2n_f)$ flavor group and, in contrast to the situation in $SU(n_c)$ theories, cannot “break up” into quarks in the vector representation. They are therefore likely to persist at the singularity and in general make the theory superconformal. We indeed checked that there are mutually non-local degrees of freedom using monodromy of the curve for $USp(4)$ theory with $n_f = 5$.¹⁸ We believe that the situation with higher n_f is also non-local; this is because a local effective low-energy Lagrangian of massless monopoles in the spinor representation of the flavor group would have an accidental $U(2^{n_f-1})$ symmetry and would lead to an unacceptably large number of Nambu–Goldstone multiplets. Once the quark masses are turned on, however, the flavor group reduces to (at least) $U(n_f)$ and it becomes possible for monopoles to break up into quarks; this explains the behavior in the equal mass case.

As for the second group of vacua, the situation is more analogous to the case of $SU(n_c)$ theories. The superpotential reads in this case (by adding mass terms to Eq.(5.10) of [18]):

$$\begin{aligned}
W = & \mu \left(\text{Tr } \phi^2 + \Lambda \sum_{a=1}^{2n_c+2-n_f} x_a \psi_a \right) + \frac{1}{\sqrt{2}} q_a^i \phi_b^a q_c^i J^{bc} + \frac{m_{ij}}{2} q_a^i q_b^j J^{ab} \\
& + \sum_{a=1}^{2n_c+2-n_f} (\psi_a e_a \tilde{e}_a + S_a^i m_i e_a \tilde{e}_a) ,
\end{aligned} \tag{8.40}$$

where $J = i\sigma_2 \otimes \mathbf{1}_{n_c}$ and

$$m = -i\sigma_2 \otimes \text{diag}(m_1, m_2, \dots, m_{n_f}) . \tag{8.41}$$

¹⁸The case $n_f = 3$, however, is special. In this case, the analysis of the curve in Section 9.2 tells us that the theory at the Chebyshev vacuum is the same as the $SU(2)$ theory with $n_f = 3$. Seiberg and Witten studied this case [2] and found that the singularity can be described in terms of massless monopoles in the spinor representation of the flavor $SO(6)$ group interacting locally with the magnetic photon. Note that there is no problem with unwanted Nambu–Goldstone multiplets in this case (see Section 8 in [2]).

By minimizing the potential, we find

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c} (\tilde{n}_c - r + 1)_{n_f} C_r \quad (8.42)$$

vacua, which precisely matches the number of the vacua of the second group, with squark VEVs behaving as

$$q_i, \tilde{q}_i \sim \sqrt{\mu m_i} \xrightarrow{m_i \rightarrow 0} 0. \quad (8.43)$$

These are the desired $SO(2n_f)$ symmetric vacua.

9 Quark-Mass Perturbation on the Curve

In this section we develop a perturbation theory in the quark masses for the singularities of the Seiberg-Witten curves of $SU(n_c)$ and $USp(2n_c)$ theories, around certain conformal points. The results of this section allows us, on the one hand, to establish the connection between the classes of CFT singularities of $N = 2$ space of vacua and the $N = 1$ vacua surviving the perturbation $\mu\Phi^2$ (as discussed in Section 6), and on the other, to identify the $N = 1$ vacua at small μ (whose physics was discussed in the previous section) with those found at large μ .

9.1 Perturbation around CFT Points: $SU(n_c)$

i) Generic r ($r < \frac{n_f}{2}$) and formulation of the problem.

Suppose the conformal point (Eq.(6.13)), $\text{diag } \phi = (\underbrace{0, 0, \dots, 0}_r, \phi_{r+1}^{(0)}, \dots, \phi_{n_c}^{(0)})$, $\sum_{a=r}^{n_c} \phi_a^{(0)} = 0$,

where the full curve with $m_i = 0$

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda^{2n_c - n_f} \prod_{j=1}^{n_f} (x + m_j)|_{m_i=0} \quad (9.1)$$

takes the form

$$y^2 = x^{2r} (x - \beta_{01})^2 \cdots (x - \beta_{0, n_c - r - 1})^2 (x - \gamma_0)(x - \kappa_0), \quad r = 0, 1, 2, \dots, [n_f/2] \quad (9.2)$$

is given. For nonzero and small bare quark masses m_i the multiple zero at the origin will split, and other zeros will be shifted. We require that the perturbed singularity,

$$\text{diag } \phi = (\lambda_1, \lambda_2, \dots, \lambda_r, \phi_{r+1}^{(0)} + \delta\phi_{r+1}, \dots, \phi_{n_c}^{(0)} + \delta\phi_{n_c}), \quad (9.3)$$

$$\sum_{a=1}^r \lambda_a + \sum_{a=r+1}^{n_c} \delta\phi_a = 0, \quad (9.4)$$

be such that the curve

$$y^2 = F(x) = \prod_{a=1}^r (x - \lambda_a)^2 \prod_{a=r+1}^{n_c} (x - \phi_{0a} - \delta\phi_a)^2 - \prod_{i=1}^{n_f} (x + m_i), \quad (9.5)$$

is maximally singular, i.e.,

$$y^2 = \prod_{a=1}^r (x - \alpha_a)^2 \left\{ \prod_{b=1}^{n_c - r - 1} (x - \beta_b)^2 \right\} (x - \gamma)(x - \kappa); \quad (9.6)$$

$$\beta_{0i} = \beta_{0i} + \delta\beta_i; \quad \gamma = \gamma_0 + \delta\gamma, \quad \kappa = \kappa_0 + \delta\kappa. \quad (9.7)$$

The problem is to determine how many such sets $\{\lambda_i, \alpha_i, \delta\beta_i, \delta\phi_i, \delta\gamma, \delta\kappa\}$ exist. The condition that the curve is maximally singular (maximal number of double branch points) can be expressed as ($F' \equiv dF(x)/dx$):

$$\begin{aligned} F(\alpha_a) &= F'(\alpha_a) = 0, \quad a = 1, 2, \dots, r; & F(\gamma) &= F(\kappa) = 0 \\ F(\beta_i) &= F'(\beta_i) = 0, \quad i = 1, 2, \dots, n_c - r - 1, \end{aligned} \quad (9.8)$$

There are $2n_c$ relations for $2n_c$ unknowns.

Let us consider the case $r = 1$ first. In this case there is only one set of $\{\alpha, \lambda\}$. Consider first the $2n_c - 2$ relations

$$F(\beta_i) = F'(\beta_i) = 0, \quad i = 1, 2, \dots, n_c - 2, \quad F(\gamma) = F(\kappa) = 0, \quad (9.9)$$

and expand each equation in the small quantities $\eta_A \equiv \{\lambda, \alpha, \delta\beta_i, \delta\phi_i, \delta\gamma, \delta\kappa\}$. By assumption these quantities are zero if $m_i = 0$. To zeroth order, (9.9) is satisfied by assumption. To first order we find a *linear* system of $2n_c - 2$ equations,

$$\begin{aligned} \sum_A \frac{dF(\beta_i)}{d\eta_A} \eta_A &= \sum_A \frac{dF'(\beta_i)}{d\eta_A} \eta_A = 0, \quad i = 1, 2, \dots, n_c - 2, \\ \sum_A \frac{dF(\gamma)}{d\eta_A} \eta_A &= \sum_A \frac{dF(\kappa)}{d\eta_A} \eta_A = 0, \end{aligned} \quad (9.10)$$

which determine $\delta\beta_i, \delta\phi_i, \delta\gamma, \delta\kappa$ uniquely, in terms of m_i 's and of λ, α .

The two equations

$$F(\alpha) = F'(\alpha) = 0 \quad (9.11)$$

(which have no zeroth order counterpart) cannot be linearized. In fact, they give

$$C(\alpha - \lambda)^2 - \prod_{i=1}^{n_f} (\alpha + m_i) = 0, \quad C = O(1); \quad (9.12)$$

$$2C(\alpha - \lambda) + C'(\alpha - \lambda)^2 - \sum_{i=1}^{n_f} \prod_{j \neq i}^{n_f} (\alpha + m_j) = 0, \quad C' = O(1). \quad (9.13)$$

Eq.(9.12) and Eq.(9.13) must be solved for α, λ .

We see immediately that α must be very special. Set

$$m_i = O(\epsilon) \ll 1, \quad (9.14)$$

and suppose $\alpha \sim O(\epsilon')$. In either cases, $\epsilon \gg \epsilon'$ or $\epsilon \ll \epsilon'$, we get from Eq.(9.12) and Eq.(9.13)

$$|(\alpha - \lambda)^2| \sim \epsilon^{n_f}; \quad |\alpha - \lambda| \sim \epsilon^{n_f-1}; \quad (9.15)$$

or an analogous relation with $\epsilon \rightarrow \epsilon'$. These cannot be satisfied (in other words, there are no solutions of this type). The only way out (to get solutions) is to assume that $\epsilon \sim \epsilon'$ and take

$$\alpha = -m_i + \Delta, \quad \Delta \leq O(\epsilon^2), \quad i = 1, 2, \dots, n_f. \quad (9.16)$$

There are obviously n_f such possibilities. We find now from Eq.(9.12) and Eq.(9.13),

$$|(\alpha - \lambda)^2| \sim \Delta \cdot \epsilon^{n_f-1}; \quad |\alpha - \lambda| \sim \epsilon^{n_f-1}. \quad (9.17)$$

Note that in Eq.(9.13) the terms containing Δ are indeed smaller than the term kept. Now these can be solved and give

$$\Delta \sim \epsilon^{n_f-1}; \quad |\alpha - \lambda| \sim \epsilon^{n_f-1}, \quad (9.18)$$

in other words,

$$\alpha = -m_i + O(\epsilon^{n_f-1}); \quad \lambda = -m_i + O(\epsilon^{n_f-1}). \quad (9.19)$$

We found obviously n_f solutions with $r = 1$, according to which m_i is used to make the solution.

A straightforward generalization to generic r , $r < \frac{n_f}{2}$ leads to the result that $\lambda_a \simeq \alpha_a$, $a = 1, 2, \dots, r$ must be chosen to be equal to r out of n_f masses, m_i . There are thus $n_f C_r$ solutions of this type, according to which masses are chosen to construct the solution. In order to see the order of magnitude of $\lambda_a - \alpha_a$, let us write the low-energy curve (at $x \ll \Lambda$) as

$$y^2 \equiv F(x) = \prod_{a=1}^r (x - \lambda_a)^2 + \frac{4}{\Lambda^{n_f-2r}} \prod_{i=1}^{n_f} (x + m_i), \quad (9.20)$$

and require that this curve behaves as

$$F(x) = \prod_{a=1}^r (x - \alpha_a)^2, \quad (9.21)$$

if we neglect zeros at $x \sim O(\Lambda)$. Namely, we require

$$F(\alpha_1) = \prod_{a=1}^r (\alpha_1 - \lambda_a)^2 + \frac{4}{\Lambda^{n_f-2r}} \prod_{i=1}^{n_f} (\alpha_1 + m_i) = 0, \quad (9.22)$$

$$F'(\alpha_1) = 2 \sum_{b=1}^r (\alpha_1 - \lambda_b) \prod_{a \neq b} (\alpha_1 - \lambda_a)^2 + \frac{4}{\Lambda^{n_f-2r}} \sum_{j=1}^{n_f} \prod_{i \neq j} (\alpha_1 + m_i) = 0, \quad (9.23)$$

and similarly for $\alpha_2, \dots, \alpha_r$. To satisfy the first equation (9.22) down to $O(m^{n_f+1})$, all we need is to set $\alpha_1 = \lambda_1$, etc. The second equation (9.23) is then approximated by keeping only the first power in $(\alpha_1 - \lambda_1)$ (in other words, only keeping $b = 1$ in the sum),

$$F'(\alpha_1) = 2(\alpha_1 - \lambda_1) \prod_{a \neq 1} (\alpha_1 - \lambda_a)^2 + \frac{4}{\Lambda^{n_f-2r}} \sum_{j=1}^{n_f} \prod_{i \neq j} (\alpha_1 + m_i) = 0. \quad (9.24)$$

Barring an accidental cancellations between α_1 and other λ_a ($a \neq 1$), we find

$$\alpha_1 - \lambda_1 \sim O(m^{n_f-1} / \Lambda^{n_f-2n_c} / m^{2(r-1)}) = O(m^{n_f-2r+1} / \Lambda^{n_f-2r}). \quad (9.25)$$

Using this fact, the first term in (9.22) is $O(m^{n_f-2r+1} / \Lambda^{n_f-2r})^2 \times O(m^{2(r-1)})$, which is much smaller than the second term of $O(m^{n_f})$. Therefore, the second term must vanish by itself, which requires $\alpha_1 = -m_1$ etc. Repeating the same analysis for every α_a , we need to choose r masses out of n_f and assign $\alpha_a = -m_a$ etc. Then we should retain only the term $j = 1$ in the second term in Eq. (9.24), and we find

$$\alpha_1 - \lambda_1 = \frac{4}{\Lambda^{n_f-2r}} \prod_{i \neq 1}^{n_f} (-m_1 + m_i) \frac{1}{2 \prod_{a \neq 1}^r (m_1 - m_a)^2}. \quad (9.26)$$

Obviously the case of equal masses is singular and beyond the validity of this analysis.

ii) $r = \frac{n_f}{2}$ with odd $n_c - n_f/2$

For special cases with $r = n_f/2$ some of the considerations above are not valid (e.g., Eq.(9.25)), and the analysis must be done ad hoc. Fortunately, in these cases it is possible to find the unperturbed configuration $\{\phi_0\}$ explicitly as in Eq.(6.16), Eq.(6.17). When n_f is even, choose $n_f/2$ eigenvalues $\phi_1 = \dots = \phi_{n_f/2}$ vanishing. Then the curve becomes

$$y^2 = x^{n_f} \left[\prod_{k=1}^{n_c - n_f/2} (x - \phi_k)^2 - 4\Lambda^{2n_c - n_f} \right]. \quad (9.27)$$

By identifying the first term in the square bracket with $(2\Lambda^{n_c - n_f/2} T_{n_c - n_f/2}(x/2\Lambda))^2$, where

$$\phi_k = 2\Lambda \cos \pi(2k - 1)/2(n_c - n_f/2), \quad k = 1, \dots, n_c - n_f/2, \quad (9.28)$$

one easily finds that

$$\begin{aligned} y^2 &= x^{n_f} \left[\left(2\Lambda^{n_c - n_f/2} T_{n_c - n_f/2} \left(\frac{x}{2\Lambda} \right) \right)^2 - 4\Lambda^{2n_c - n_f} \right] \\ &= -4x^{n_f} \Lambda^{2n_c - n_f} \sin^2 \left[\left(n_c - \frac{n_f}{2} \right) \arccos \frac{x}{2\Lambda} \right]. \end{aligned} \quad (9.29)$$

There are n_f zeros at $x = 0$, and double zeros at $x = 2\Lambda \cos \pi k / (n_c - n_f/2)$ for $k = 1, \dots, n_c - n_f/2 - 1$, and single zeros at $x = \pm 2\Lambda$.

In the absence of quark masses, the theory is invariant under $Z_{2n_c - n_f}$ symmetry: $x \rightarrow e^{2\pi i/(2n_c - n_f)} x$, $\phi_a \rightarrow e^{2\pi i/(2n_c - n_f)} \phi_a$. However, the gauge invariant symmetric polynomials of ϕ_k vanish for odd powers because of the equal number of positive and negative ones with the same absolute values. Therefore the Chebyshev solutions discussed here appear only $n_c - n_f/2$ times. This point is crucial for the vacuum counting (see Eq.(6.21)) to work out correctly.

We first study the specific case of $2n_c = n_f + 2$. More general cases will be seen to reduce to this case. The Chebyshev solution is obtained in the massless limit by setting all of $\phi_a = 0$:

$$y^2 = x^{2n_c} - 4\Lambda^2 x^{n_f} = x^{n_f} (x + 2\Lambda)(x - 2\Lambda). \quad (9.30)$$

The zero at $x = 0$ is of degree n_f , and there are other isolated zeros at $x = \pm 2\Lambda$. Under the perturbation by generic quark masses, we go back to the original curve. The only way that the curve can be arranged to have $n_c - 1$ double zeros as

$$y^2 = \prod_{a=1}^{n_c - 1} (x - \alpha_a)^2 (x + 2\Lambda - \beta)(x - 2\Lambda - \gamma) \quad (9.31)$$

is by assuming

$$\alpha_a \sim m, \quad \phi_1 \sim \dots \sim \phi_{n_c - 2} \sim m, \quad \phi_{n_c - 1} = -\phi_{n_c} \sim (m\Lambda)^{1/2}. \quad (9.32)$$

(These behaviors have been suggested by the numerical solution of several explicit examples.) Studying the region $x \sim m$, and neglecting $\beta, \gamma \ll \Lambda$, $x \ll \phi_{n_c}$, we need to solve the equation

$$\phi_{n_c}^4 \prod_{a=1}^{n_c - 2} (x - \phi_a)^2 - 4\Lambda^2 \prod_{i=1}^{2n_c - 2} (x + m_i) = -4\Lambda^2 \prod_{a=1}^{n_c - 1} (x - \alpha_a)^2. \quad (9.33)$$

Note that the traceless condition for ϕ_a is not a stringent constraint because $\phi_{n_c-1} = -\phi_{n_c} = O(m\Lambda)^{1/2}$ can shift by small amount to absorb the trace of ϕ_a . By moving terms around, we find

$$\begin{aligned} & \prod_{i=1}^{2n_c-2} (x + m_i) \\ &= \left[\prod_{a=1}^{n_c-1} (x - \alpha_a) + i \frac{\phi_{n_c}^2}{2\Lambda} \prod_{a=1}^{n_c-2} (x - \phi_a) \right] \left[\prod_{a=1}^{n_c-1} (x - \alpha_a) - i \frac{\phi_{n_c}^2}{2\Lambda} \prod_{a=1}^{n_c-2} (x - \phi_a) \right]. \end{aligned} \quad (9.34)$$

For this identity to hold for any x , $2n_c - 2$ zeros of l.h.s. $(-m_i)$ must coincide with $n_c - 2$ zeros in each of square brackets in the r.h.s. of the equation. Therefore, we divide up $2n_c - 2$ masses into two sets $\{p_i, i = 1, \dots, n_c - 1\}$ and $\{q_i, i = 1, \dots, n_c - 1\}$. This gives us $n_f C_{n_f/2}$ choices. By identifying p_i to the first square bracket and q_i to the second square bracket, we find

$$\begin{aligned} \prod_{i=1}^{n_c-1} (x + p_i) &= \prod_{a=1}^{n_c-1} (x - \alpha_a) + i \frac{\phi_{n_c}^2}{2\Lambda} \prod_{a=1}^{n_c-2} (x - \phi_a), \\ \prod_{i=1}^{n_c-1} (x + q_i) &= \prod_{a=1}^{n_c-1} (x - \alpha_a) - i \frac{\phi_{n_c}^2}{2\Lambda} \prod_{a=1}^{n_c-2} (x - \phi_a). \end{aligned} \quad (9.35)$$

Expanding both sides in terms of symmetric polynomials,

$$\begin{aligned} \sum_{k=0}^{n_c-1} s_k(p) x^{n_c-1-k} &= \sum_{k=0}^{n_c-1} (-1)^k s_k(\alpha) x^{n_c-k-1} + i \frac{\phi_{n_c}^2}{2\Lambda} \sum_{k=0}^{n_c-2} (-1)^k s_k(\phi) x^{n_c-k-2}, \\ \sum_{k=0}^{n_c-1} s_k(q) x^{n_c-1-k} &= \sum_{k=0}^{n_c-1} (-1)^k s_k(\alpha) x^{n_c-k-1} - i \frac{\phi_{n_c}^2}{2\Lambda} \sum_{k=0}^{n_c-2} (-1)^k s_k(\phi) x^{n_c-k-2}. \end{aligned} \quad (9.36)$$

Finally, we find the solutions

$$\begin{aligned} (-1)^k s_k(p) &= s_k(\alpha) - i \frac{\phi_{n_c}^2}{2\Lambda} s_{k-1}(\phi), \\ (-1)^k s_k(q) &= s_k(\alpha) + i \frac{\phi_{n_c}^2}{2\Lambda} s_{k-1}(\phi). \end{aligned} \quad (9.37)$$

Here and below, we use the notation that $s_{-1} = 0$. By setting $k = 1$ and subtracting both sides, we find $\phi_{n_c}^2 = -i\Lambda(s_1(p) - s_1(q)) = -i\Lambda \sum_{k=1}^{n_f/2} (p_i - q_i)$. All $s_k(\alpha)$ and $s_{k-1}(\phi)$ are then given in terms of $s_k(p)$ and $s_k(q)$, and hence we can write order $n_c - 1$ ($n_c - 2$) polynomial equation for α (ϕ) which can always be solved to find all α 's (ϕ 's).

For smaller even n_f with $n_c - n_f/2$ odd, the problem reduces the one with $2n_c - n_f = 2$. To see this, we write the curve around the Chebyshev point as

$$y^2 = \prod_{a=1}^{n_f/2} (x - \phi_a)^2 \prod_{k=1}^{n_c-n_f/2} (x - \phi_k)^2 - 4\Lambda^{2n_c-n_f} \prod_{i=1}^{n_c-n_f/2} (x + m_i), \quad (9.38)$$

where $\phi_k = 2\Lambda \cos \pi(k - 1/2)/(n_c - n_f/2)$. When $k = (n_c - n_f/2 + 1)/2$, $\phi_k = 0$. Therefore, neglecting the fluctuations of ϕ_k ($k \neq (n_c - n_f/2 + 1)/2$), the curve reduces to

$$y^2 = \prod_{a=1}^{n_f/2+1} (x - \phi_a)^2 \prod_{k=1, k \neq (n_c - n_f/2 + 1)/2}^{n_c - n_f/2} \phi_k^2 - 4\Lambda^{2n_c - n_f} \prod_{i=1} (x + m_i), \quad (9.39)$$

where ϕ_a with $a = n_f/2 + 1$ is ϕ_k with $k = (n_c - n_f/2 + 1)/2$. The product of non-vanishing ϕ_k 's can be obtained as follows. Recalling $T_N(x) = 2^{N-1} \prod_{k=1}^N (x - w_k)$ with $w_k = \cos \pi(k - 1/2)/N$, we obtain for odd N , $T'_N(0) = 2^{N-1} \prod_{k \neq (n_c - n_f/2 + 1)/2} (-w_k)$, while $T'_N(x) = N \sin(N \arccos x) / \sqrt{1 - x^2}$ and hence $T'_N(0) = N \sin N\pi/2 = N(-1)^{(N-1)/2}$. Therefore, $\prod_{k=1, k \neq (n_c - n_f/2 + 1)/2}^{n_c - n_f/2} \phi_k^2 = (T'_N(0))^2 \Lambda^{2N-2} = N^2 \Lambda^{2N-2}$. The low-energy curve is then

$$\frac{y^2}{(n_c - n_f/2)^2 \Lambda^{2n_c - n_f - 2}} = \prod_{a=1}^{n_f/2+1} (x - \phi_a)^2 - 4 \frac{\Lambda^2}{(n_c - n_f/2)^2} \prod_{i=1} (x + m_i). \quad (9.40)$$

This is nothing but the curve of $SU(n_f/2 + 1)$ theory with the dynamical scale $\Lambda/(n_c - n_f/2)$.

iii) $r = \frac{n_f}{2}$ with even $n_c - n_f/2$

The Chebyshev solution Eq.(9.28), Eq.(9.29) is valid in this case also. Again, we first study the special case with $2n_c = n_f + 4$. The Chebyshev solution is obtained in the massless limit by setting all but two of $\phi_a = 0$:

$$y^2 = x^{n_f} [(x - \phi_{n_c-1})^2 (x - \phi_{n_c})^2 - 4\Lambda^4] = x^{n_f} (x^2 - \phi_{n_c}^2 - 2\Lambda^2)(x^2 - \phi_{n_c}^2 + 2\Lambda^2). \quad (9.41)$$

By choosing $-\phi_{n_c-1} = \phi_{n_c} = \sqrt{2}\Lambda$, the curve becomes

$$y^2 = x^{2n_c} - 4\Lambda^2 x^{n_f+2} = x^{n_f+2} (x - 2\Lambda)(x + 2\Lambda). \quad (9.42)$$

The zero at $x = 0$ is of degree $n_f + 2$, and there are other isolated zeros at $x = \pm 2\Lambda$. There is another solution with $-\phi_{n_c-1} = \phi_{n_c} = i\sqrt{2}\Lambda$ as required by the discrete Z_4 symmetry under which the Chebyshev solution transforms as a doublet.

Under the perturbation by generic quark masses, we go back to the original curve. The only way that the curve can be arranged to have $n_c - 1$ double zeros as

$$y^2 = \prod_{a=1}^{n_c-1} (x - \alpha_a)^2 (x + 2\Lambda - \beta)(x - 2\Lambda - \gamma) \quad (9.43)$$

is by assuming

$$\phi_a \sim m, \quad \alpha_1 \sim \dots \sim \alpha_{n_c-2} \sim m, \quad \alpha_{n_c-2} = -\alpha_{n_c-1} \sim (m\Lambda)^{1/2}. \quad (9.44)$$

(The choice $\alpha_{n_c-1} = -\alpha_{n_c}$ has been suggested from several explicit examples studied numerically.) Studying the region $x \sim m$, and neglecting $\beta, \gamma \ll \Lambda$, $x \ll \alpha_{n_c-1}$, we must solve the equation

$$4\Lambda^4 \prod_{a=1}^{n_c-2} (x - \phi_a)^2 - 4\Lambda^4 \prod_{i=1}^{2n_c-4} (x + m_i) = -4\Lambda^2 \alpha_{n_c-1}^4 \prod_{a=1}^{n_c-3} (x - \alpha_a)^2. \quad (9.45)$$

Note that the traceless condition for ϕ_a is not a stringent constraint because $\phi_{n_c-1} = -\phi_{n_c} = O(\Lambda)$ can shift by small amount to absorb the trace of ϕ_a . By moving terms around, we find

$$\begin{aligned} & \prod_{i=1}^{2n_c-4} (x + m_i) \\ &= \left[\prod_{a=1}^{n_c-2} (x - \phi_a) + i \frac{\alpha_{n_c-1}^2}{\Lambda} \prod_{a=1}^{n_c-3} (x - \alpha_a) \right] \left[\prod_{a=1}^{n_c-2} (x - \phi_a) - i \frac{\alpha_{n_c-1}^2}{\Lambda} \prod_{a=1}^{n_c-3} (x - \alpha_a) \right]. \end{aligned} \quad (9.46)$$

For this identity to hold for any x , $2n_c - 4$ zeros of l.h.s. ($-m_i$) must coincide with $n_c - 1$ zeros in each of square brackets in the r.h.s. of the equation. Therefore, we divide up $2n_c - 4 = n_f$ masses into two sets $\{p_i, i = 1, \dots, n_c - 2\}$ and $\{q_i, i = 1, \dots, n_c - 2\}$. This gives us $n_f C_{n_f/2}$ choices. By identifying p_i to the first square bracket and q_i to the second square bracket, we find

$$\begin{aligned} \prod_{i=1}^{n_c-2} (x + p_i) &= \prod_{a=1}^{n_c-2} (x - \phi_a) + i \frac{\alpha_{n_c-1}^2}{\Lambda} \prod_{a=1}^{n_c-3} (x - \alpha_a), \\ \prod_{i=1}^{n_c-1} (x + q_i) &= \prod_{a=1}^{n_c-2} (x - \phi_a) - i \frac{\alpha_{n_c-1}^2}{\Lambda} \prod_{a=1}^{n_c-3} (x - \alpha_a). \end{aligned} \quad (9.47)$$

Expanding both sides in terms of symmetric polynomials,

$$\begin{aligned} \sum_{k=0}^{n_c-2} s_k(p) x^{n_c-2-k} &= \sum_{k=0}^{n_c-2} (-1)^k s_k(\phi) x^{n_c-k-2} + i \frac{\alpha_{n_c-1}^2}{\Lambda} \sum_{k=0}^{n_c-3} (-1)^k s_k(\alpha) x^{n_c-k-2}, \\ \sum_{k=0}^{n_c-1} s_k(q) x^{n_c-1-k} &= \sum_{k=0}^{n_c-2} (-1)^k s_k(\phi) x^{n_c-k-2} - i \frac{\alpha_{n_c-1}^2}{\Lambda} \sum_{k=0}^{n_c-3} (-1)^k s_k(\alpha) x^{n_c-k-2}. \end{aligned} \quad (9.48)$$

Finally, we find the solutions

$$(-1)^k s_k(p) = s_k(\phi) - i \frac{\alpha_{n_c-1}^2}{\Lambda} s_{k-1}(\alpha), \quad (-1)^k s_k(q) = s_k(\phi) + i \frac{\alpha_{n_c-1}^2}{\Lambda} s_{k-1}(\alpha). \quad (9.49)$$

Here and below, we use the notation that $s_{-1} = 0$ identically. By setting $k = 1$ and subtracting both sides, we find $\alpha_{n_c-1}^2 = -i\Lambda(s_1(p) - s_1(q))/2 = -i\Lambda \sum_{k=1}^{n_f/2} (p_i - q_i)/2$. All $s_k(\phi)$ and $s_{k-1}(\alpha)$ are then given in terms of $s_k(p)$ and $s_k(q)$, and hence we can write order $n_c - 3$ ($n_c - 2$) polynomial equation for α (ϕ) which can always be solved to find all α 's (ϕ 's).

For smaller even n_f with $n_c - n_f/2$ even, the curve reduces to the one with $2n_c - n_f = 4$. To see this, we write the curve around the Chebyshev point as

$$y^2 = \prod_{a=1}^{n_f/2} (x - \phi_a)^2 \prod_{k=1}^{n_c-n_f/2} (x - \phi_k)^2 - 4\Lambda^{2n_c-n_f} \prod_{i=1}^{n_c-n_f/2} (x + m_i), \quad (9.50)$$

where $\phi_k = 2\Lambda \cos \pi(k - 1/2)/(n_c - n_f/2)$. Therefore, neglecting the fluctuations of ϕ_k ($k \neq (n_c - n_f/2 + 1)/2$), the curve reduces to

$$y^2 = \prod_{a=1}^{n_f/2} (x - \phi_a)^2 \prod_{k=1}^{n_c - n_f/2} \phi_k^2 - 4\Lambda^{2n_c - n_f} \prod_{i=1}^{n_f/2} (x + m_i). \quad (9.51)$$

The product of ϕ_k 's can be obtained as follows. Recalling $T_N(x) = 2^{N-1} \prod_{k=1}^N (x - w_k)$ with $w_k = \cos \pi(k - 1/2)/N$, we obtain for even N , $T_N(0) = 2^{N-1} \prod_k (-w_k)$, while $T_N(x) = \cos(N \arccos x)$ and hence $T_N(0) = \cos N\pi/2 = (-1)^{N/2}$. Therefore, $\prod_{k=1}^{n_c - n_f/2} \phi_k^2 = (2T_N(0))^2 \cdot \Lambda^{2N} = 4\Lambda^{2N}$. Therefore the low-energy curve is

$$\frac{y^2}{4\Lambda^{2n_c - n_f}} = \prod_{a=1}^{n_f/2} (x - \phi_a)^2 - \prod_{i=1}^{n_f/2} (x + m_i). \quad (9.52)$$

Note that this is precisely the curve of the $SU(\frac{n_f}{2})$ theory. It has the same form as the left hand side of Eq.(9.45).

iv) $r = \tilde{n}_c = n_f - n_c$: Root of the baryonic branch

The case with $r = \tilde{n}_c = n_f - n_c$ also requires a separate consideration since the unperturbed curve has a special form, Eq.(6.18), Eq.(6.19). The curve is

$$y^2 = x^{2\tilde{n}_c} \left[\prod_{k=1}^{n_c - \tilde{n}_c} (x - \Phi_k)^2 + 4\Lambda^{2n_c - n_f} x^{n_c - \tilde{n}_c} \right] = x^{2\tilde{n}_c} (x^{n_c - \tilde{n}_c} - \Lambda^2)^2. \quad (9.53)$$

with adjoint VEVs taken as $\phi_1, \dots, \phi_{n_c - \tilde{n}_c} \neq 0$, $\phi_{n_c - \tilde{n}_c + 1} = \dots = \phi_{n_c} = 0$,

$$(\Phi_1, \dots, \Phi_k, \dots, \Phi_{n_c - \tilde{n}_c}) = (\Lambda\omega^2, \dots, \omega^{2k}, \dots, \omega^{2(n_c - \tilde{n}_c)}), \quad (9.54)$$

where $\omega = e^{\pi i/(n_c - \tilde{n}_c)}$. The double zeros of the factor in the parenthesis are at

$$x = \Lambda\omega, \Lambda\omega^3, \dots, \Lambda\omega^{2k-1}, \dots, \Lambda\omega^{2(n_c - \tilde{n}_c) - 1}. \quad (9.55)$$

There are two ways for maintaining the curve maximally singular, when generic bare quark masses are added. One is to keep all “large” $n_c - \tilde{n}_c$ zeros doubled, while allowing $2\tilde{n}_c$ zeros at $x = 0$ to decompose into $\tilde{n}_c - 1$ double zeros and two single zeros. The other is to take all “small” zeros doubled, while keeping only $n_c - \tilde{n}_c - 1$ “large” double zeros.

iv-a) Keeping all of “large” zeros doubled

Upon mass perturbation the adjoint scalar VEVs take the form, $(\phi_a, \Phi_k = \Lambda\omega^{2k} + \gamma_k)$, with $\gamma_k, \phi_a \sim O(m)$. The constraint is therefore

$$\sum_{a=1}^{\tilde{n}_c} \phi_a + \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k = 0. \quad (9.56)$$

The perturbed curve is

$$y^2 = \prod_{a=1}^{\tilde{n}_c} (x - \phi_a)^2 \prod_{k=1}^{n_c - \tilde{n}_c} (x - \Lambda\omega^{2k} - \gamma_k)^2 + 4\Lambda^{2n_c - n_f} \prod_{i=1}^{n_f/2} (x + m_i). \quad (9.57)$$

The zeros (9.55) are also shifted to

$$x = \Lambda\omega^{2\ell-1} + \delta_\ell, \quad \ell = 1, \dots, n_c - \tilde{n}_c. \quad (9.58)$$

We substitute these zeros for each ℓ into the curve and require that the r.h.s. of the curve (9.57) vanishes at $O(m)$.

The first factor in the curve (9.57) is expanded as

$$\prod_{a=1}^{\tilde{n}_c} (\Lambda\omega^{2\ell-1} + \delta_\ell - \phi_a)^2 = (\Lambda\omega^{2\ell-1})^{\tilde{n}_c} \left[1 + 2 \sum_{a=1}^{\tilde{n}_c} \frac{\delta_\ell - \phi_a}{\Lambda\omega^{2\ell-1}} \right]. \quad (9.59)$$

The second factor in the curve (9.57) is expanded as

$$\prod_{k=1}^{n_c - \tilde{n}_c} (\Lambda\omega^{2\ell-1} + \delta_\ell - \Lambda\omega^{2k} - \gamma_k)^2 = \prod_{k=1}^{n_c - \tilde{n}_c} (\Lambda\omega^{2\ell-1} - \Lambda\omega^{2k})^2 \left[1 + 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\delta_\ell - \gamma_k}{\Lambda\omega^{2\ell-1} - \Lambda\omega^{2k}} \right]. \quad (9.60)$$

This factor needs to be simplified. We show that

$$\prod_{k=1}^{n_c - \tilde{n}_c} (\Lambda\omega^{2\ell-1} - \Lambda\omega^{2k}) = -2\Lambda^{n_c - \tilde{n}_c}. \quad (9.61)$$

This can be proven by studying the polynomial

$$\prod_{k=1}^{n_c - \tilde{n}_c} (t - \Lambda\omega^{2k}) = t^{n_c - \tilde{n}_c} - \Lambda^{n_c - \tilde{n}_c}. \quad (9.62)$$

By substituting $t = \Lambda\omega^{2\ell-1}$, and noting that $(\Lambda\omega^{2\ell-1})^{n_c - \tilde{n}_c} = \Lambda^{n_c - \tilde{n}_c} e^{\pi i(2\ell-1)} = -\Lambda^{n_c - \tilde{n}_c}$. The second factor (9.60) of the curve (9.57) is now simplified to

$$4\Lambda^{2(n_c - \tilde{n}_c)} \left[1 + 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\delta_\ell - \gamma_k}{\Lambda\omega^{2\ell-1} - \Lambda\omega^{2k}} \right]. \quad (9.63)$$

The last term in the curve (9.57) is expanded as

$$4\Lambda^{2n_c - n_f} \prod_{i=1}^{n_f} (\Lambda\omega^{2\ell-1} + \delta_\ell + m_i) = 4\Lambda^{2n_c} (\omega^{2\ell-1})^{n_f} \left[1 + \sum_{i=1}^{n_f} \frac{\delta_\ell + m_i}{\Lambda\omega^{2\ell-1}} \right]. \quad (9.64)$$

Note further that

$$(\omega^{2\ell-1})^{n_f} = (\omega^{2\ell-1})^{2\tilde{n}_c + (n_c - \tilde{n}_c)} = (\omega^{2\ell-1})^{2\tilde{n}_c} e^{\pi i(2\ell-1)} = -(\omega^{2\ell-1})^{2\tilde{n}_c}. \quad (9.65)$$

The last term therefore is

$$-4\Lambda^{2n_c} (\omega^{2\ell-1})^{2\tilde{n}_c} \left[1 + \sum_{i=1}^{n_f} \frac{\delta_\ell + m_i}{\Lambda\omega^{2\ell-1}} \right]. \quad (9.66)$$

By putting together the expansion Eqs.(9.59,9.63,9.66) up to $O(m)$ into the curve (9.57) and requiring it to vanish, we find

$$0 = 2 \sum_{a=1}^{\tilde{n}_c} \frac{\delta_\ell - \phi_a}{\Lambda \omega^{2\ell-1}} + 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\delta_\ell - \gamma_k}{\Lambda \omega^{2\ell-1} - \Lambda \omega^{2k}} - \sum_{i=1}^{n_f} \frac{\delta_\ell + m_i}{\Lambda \omega^{2\ell-1}}. \quad (9.67)$$

Multiply the equation by $\Lambda \omega^{2\ell-1}$ to simplify it to

$$0 = 2 \sum_{a=1}^{\tilde{n}_c} (\delta_\ell - \phi_a) + 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\delta_\ell - \gamma_k}{1 - \omega^{2k-2\ell+1}} - \sum_{i=1}^{n_f} (\delta_\ell + m_i). \quad (9.68)$$

We try to further simplify this equation. The sum over a in the first term gives

$$2 \sum_{a=1}^{\tilde{n}_c} (\delta_\ell - \phi_a) = 2\tilde{n}_c \delta_\ell - 2 \sum_{a=1}^{\tilde{n}_c} \phi_a = 2\tilde{n}_c \delta_\ell + 2 \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k, \quad (9.69)$$

where the constraint Eq. (9.56) was used in the last equality. To simplify the second term, we would like to prove that

$$\sum_{k=1}^{n_c - \tilde{n}_c} \frac{1}{1 - \omega^{2k-2\ell+1}} = \frac{1}{2}(n_c - \tilde{n}_c). \quad (9.70)$$

Note that the sum over k exhausts all possible odd powers in ω in the denominator. Therefore we can shift k to $k + \ell$ and find

$$\sum_{k=1}^{n_c - \tilde{n}_c} \frac{1}{1 - \omega^{2k-2\ell+1}} = \sum_{k=1}^{n_c - \tilde{n}_c} \frac{1}{1 - \omega^{2k+1}}. \quad (9.71)$$

Now we distinguish two cases, when $n_c - \tilde{n}_c = 2m$ (even), and $n_c - \tilde{n}_c = 2m + 1$ (odd). When $n_c - \tilde{n}_c = 2m$, the sum is separated in two pieces,

$$\sum_{k=1}^{2m} \frac{1}{1 - \omega^{2k+1}} = \sum_{k=1}^m \frac{1}{1 - \omega^{2k+1}} + \sum_{k=m+1}^{2m} \frac{1}{1 - \omega^{2k+1}}. \quad (9.72)$$

Change the variable k in the second sum to $2m - k + 1$, and using the definition $\omega = e^{\pi i/2m}$,

$$= \sum_{k=1}^m \frac{1}{1 - \omega^{2k+1}} + \sum_{k=1}^m \frac{1}{1 - \omega^{-(2k+1)}}. \quad (9.73)$$

Adding terms for each m ,

$$= \sum_{k=1}^m \frac{1 - \omega^{2k+1} + 1 - \omega^{-(2k+1)}}{1 - \omega^{2k+1} - \omega^{-(2k+1)} + 1} = m = \frac{1}{2}(n_c - \tilde{n}_c). \quad (9.74)$$

This completes the proof of Eq. (9.70) for even $n_c - \tilde{n}_c$. Similarly, for $n_c - \tilde{n}_c = 2m + 1$,

$$\sum_{k=1}^{2m+1} \frac{1}{1 - \omega^{2k+1}} = \sum_{k=1}^m \frac{1}{1 - \omega^{2k+1}} + \frac{1}{1 - \omega^{2m+1}} + \sum_{k=m+2}^{2m+1} \frac{1}{1 - \omega^{2k+1}}. \quad (9.75)$$

The middle term in the r.h.s. is $1/(1 - (-1)) = 1/2$. Change the variable k in the last sum to $2m + 2 - k$,

$$= \sum_{k=1}^m \frac{1}{1 - \omega^{2k+1}} + \frac{1}{2} + \sum_{k=1}^m \frac{1}{1 - \omega^{-(2k+1)}}. \quad (9.76)$$

Again by adding terms for each m ,

$$= \frac{1}{2} + \sum_{k=1}^m \frac{1 - \omega^{2k+1} + 1 - \omega^{-(2k+1)}}{1 - \omega^{2k+1} - \omega^{-(2k+1)} + 1} = \frac{1}{2} + m = \frac{1}{2}(n_c - \tilde{n}_c). \quad (9.77)$$

This completes the proof of Eq. (9.70) for odd $n_c - \tilde{n}_c$. Therefore the identity (9.70) is proven. The second term in Eq. (9.68) is then given by

$$2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\delta_\ell - \gamma_k}{1 - \omega^{2k-2\ell+1}} = (n_c - \tilde{n}_c)\delta_\ell - 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\gamma_k}{1 - \omega^{2k-2\ell+1}}. \quad (9.78)$$

Finally the last term in Eq. (9.68) is simply

$$- \sum_{i=1}^{n_f} (\delta_\ell + m_i) = -n_f \delta_\ell - \sum_{i=1}^{n_f} m_i. \quad (9.79)$$

Putting together Eqs. (9.69,9.78,9.79) in (9.68), we find

$$\begin{aligned} 0 &= 2\tilde{n}_c \delta_\ell + 2 \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k + (n_c - \tilde{n}_c)\delta_\ell - 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\gamma_k}{1 - \omega^{2k-2\ell+1}} - n_f \delta_\ell - \sum_{i=1}^{n_f} m_i \\ &= 2 \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k - 2 \sum_{k=1}^{n_c - \tilde{n}_c} \frac{\gamma_k}{1 - \omega^{2k-2\ell+1}} - \sum_{i=1}^{n_f} m_i, \end{aligned} \quad (9.80)$$

and δ_ℓ disappeared from the equation. Finally, we add over ℓ . Only the second term depends on ℓ , and the sum over ℓ is simplified again by using the identity Eq. (9.70). We find

$$0 = 2(n_c - \tilde{n}_c) \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k - 2 \frac{1}{2}(n_c - \tilde{n}_c) \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k - (n_c - \tilde{n}_c) \sum_{i=1}^{n_f} m_i, \quad (9.81)$$

and therefore

$$\sum_{a=1}^{\tilde{n}_c} \phi_a = - \sum_{k=1}^{n_c - \tilde{n}_c} \gamma_k = - \sum_{i=1}^{n_f} m_i. \quad (9.82)$$

The fluctuation around $x \sim 0$ is described by a low-energy curve by neglecting $x, \gamma_k \ll \Lambda$ in Eq. (9.57),

$$y^2 = \prod_{a=1}^{\tilde{n}_c} (x - \phi_a)^2 \prod_{k=1}^{n_c - \tilde{n}_c} (-\Lambda \omega^{2k})^2 + 4\Lambda^{2n_c - n_f} \prod_{i=1}^{n_f} (x + m_i), \quad (9.83)$$

or

$$\frac{y^2}{\Lambda^{2(n_c - \tilde{n}_c)}} = \prod_{a=1}^{\tilde{n}_c} (x - \phi_a)^2 + \frac{4}{\Lambda^{n_c - \tilde{n}_c}} \prod_{i=1}^{n_f} (x + m_i), \quad (9.84)$$

where ϕ_a are subject to an unusual constraint Eq. (9.82). Apart from the constraint, the problem is similar to earlier study of the r -root, where we take $\phi_a = -m_a$ to have double zeros at $x = -m_a$. The maximum number of ϕ_a 's that can be chosen this way is, however, $\tilde{n}_c - 1$ because of the constraint. Let us choose r out of \tilde{n}_c to coincide with r of $-m_i$ (${}_{n_f}C_r$ choices), *e.g.*,

$$\phi_1 = -m_1, \dots, \phi_r = -m_r, \quad (9.85)$$

While the remaining ϕ_a are

$$\phi_k = -\frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i + \delta\phi_k, \quad k = 1, \dots, \tilde{n}_c - r. \quad (9.86)$$

The fluctuations are subject to the constraint

$$\sum_{k=1}^{\tilde{n}_c - r} \delta\phi_k = 0. \quad (9.87)$$

The low-energy curve Eq. (9.84) then becomes

$$\frac{y^2}{\Lambda^{2(n_c - \tilde{n}_c)}} = \prod_{a=1}^r (x - m_a)^2 \prod_{k=1}^{\tilde{n}_c - r} \left(x + \frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i - \delta\phi_k \right)^2 + \frac{4}{\Lambda^{n_c - \tilde{n}_c}} \prod_{i=1}^{n_f} (x + m_i). \quad (9.88)$$

Now shift x to $x - \frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i$, and we assume that the remaining fluctuations $x, \delta\phi_k \ll m$ which will be justified *a posteriori*. The curve is further approximated as

$$\begin{aligned} \frac{y^2}{\Lambda^{2(n_c - \tilde{n}_c)}} &= \prod_{a=1}^r \left(\frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i + m_a \right)^2 \prod_{k=1}^{\tilde{n}_c - r} (x - \delta\phi_k)^2 \\ &+ \frac{4}{\Lambda^{n_c - \tilde{n}_c}} \prod_{i=1}^{n_f} \left(m_i - \frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i \right). \end{aligned} \quad (9.89)$$

Up to an overall constant, this is the curve of pure $SU(\tilde{n}_c - r)$ Yang–Mills theories. The dynamical scale of this theory is

$$\begin{aligned} \Lambda_{pure}^{2(\tilde{n}_c - r)} &= \frac{1}{\Lambda^{n_c - \tilde{n}_c}} \frac{\prod_{i=1}^{n_f} \left(m_i - \frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i \right)}{\prod_{a=1}^r \left(\frac{1}{\tilde{n}_c - r} \sum_{i=r+1}^{n_f} m_i + m_a \right)^2} \\ &\sim m^{2(\tilde{n}_c - r)} \left(\frac{m}{\Lambda} \right)^{n_c - \tilde{n}_c} \ll m^{2(\tilde{n}_c - r)}. \end{aligned} \quad (9.90)$$

Therefore the singularities of this curve are located at $x, \delta\phi_k \sim \Lambda_{pure} \ll m$ and hence the approximation is justified. There are $\tilde{n}_c - r$ such singularities (basically the Witten index) for each r , and hence the total number of vacua obtained in this subsection are

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c - 1} (\tilde{n}_c - r) {}_{n_f}C_r. \quad (9.91)$$

iv-b) Keeping all of “small” double zeros

Another possibility is that all \tilde{n}_c “small” zeros near $x = 0$ are doubled, while one of the $n_c - \tilde{n}_c$ “large” double zeros is split into two single zeros.

The crucial difference from the previous case is that we keep one less “large” double zeros and hence fluctuations $\phi_1, \dots, \phi_{\tilde{n}_c}$ are not subject to a constraint. Even though the low-energy curve

$$\frac{y^2}{\Lambda^{2(n_c - \tilde{n}_c)}} = \prod_{a=1}^{\tilde{n}_c} (x - \phi_a)^2 + \frac{4}{\Lambda^{n_c - \tilde{n}_c}} \prod_{i=1}^{n_f} (x + m_i), \quad (9.92)$$

is the same as Eq. (9.84) in the previous subsection, ϕ_a can all freely vary and hence can all be matched to the quark masses as in the cases i) above, as

$$\phi_1 = -m_1, \dots, \phi_{\tilde{n}_c} = -m_{\tilde{n}_c}, \quad (9.93)$$

and there are $n_f C_{\tilde{n}_c}$ choices. Recall that there are $n_c - \tilde{n}_c$ “large” double zeros at the baryonic root and we can give up one of them; therefore there are actually $(n_c - \tilde{n}_c) n_f C_{\tilde{n}_c}$ choices.

v) Summary of the vacuum counting in $SU(n_c)$ theories

The “nonbaryonic” branch roots Eq.(9.2), Eq.(9.29), yield, upon quark mass perturbation,

$$(2n_c - n_f) \cdot 2^{n_f - 1} - (n_c - \tilde{n}_c) n_f C_{\tilde{n}_c} = \mathcal{N}_1 - (n_c - \tilde{n}_c) n_f C_{\tilde{n}_c} \quad (9.94)$$

vacua. The baryonic root, Eq.(9.53), Eq.(9.54), leads to

$$\sum_{r=0}^{\tilde{n}_c - 1} (\tilde{n}_c - r) n_f C_r + (n_c - \tilde{n}_c) n_f C_{\tilde{n}_c} = \mathcal{N}_2 + (n_c - \tilde{n}_c) n_f C_{\tilde{n}_c} \quad (9.95)$$

vacua. Their sum coincides with the total number of $N = 1$ vacua found from the semiclassical analysis as well as from the analyses at large μ .

9.2 Perturbation around CFT points of the $USp(2n_c)$ Curve

We start from the CFT points described by the Chebyshev polynomial, Eq.(6.26)-Eq.(6.29), and add generic quark masses m_i .

i) Chebyshev point: Odd n_f

Let us take n_f odd first. We first study the specific case of $2n_c = n_f - 1$. The more general cases will be discussed later on.

The Chebyshev solution is obtained in the massless limit by setting all of $\phi_a = 0$:

$$xy^2 = [x^{n_c + 1}]^2 - 4\Lambda^2 x^{2n_c + 1} = x^{2n_c + 1} (x - 4\Lambda^2). \quad (9.96)$$

The zero at $x = 0$ is of degree $2n_c$, and there is another isolated zero at $x = 4\Lambda^2$. There is also a branch point at $x = \infty$.

Under the perturbation by generic quark masses, we go back to the original curve. The only way that the curve can be arranged to have n_c double zeros as

$$xy^2 = x(x - 4\Lambda^2 - \beta) \prod_{a=1}^{n_c} (x - \alpha_a)^2 \quad (9.97)$$

is by assuming

$$\alpha_a \sim m^2, \quad \phi_1^2 \sim m\Lambda, \quad \phi_2^2 \sim \dots \sim \phi_{n_c}^2 \sim m^2. \quad (9.98)$$

By neglecting $\beta, x \ll \Lambda^2$, we need to solve the equation

$$\left[x \prod_{a=1}^{n_c} (x - \phi_a^2) + 2\Lambda m_1 \dots m_{n_f} \right]^2 - 4\Lambda^2 \prod_{i=1}^{n_f} (x + m_i^2) = -4\Lambda^2 x \prod_{a=1}^{n_c} (x - \alpha_a)^2. \quad (9.99)$$

To address this question, we introduce a few preliminary facts. First of all, consider a generic polynomial

$$\prod_{i=1}^N (z - \rho_i) = \sum_{k=0}^N (-1)^k s_k(\rho) z^{N-k}. \quad (9.100)$$

The symmetric polynomials $s_j(\rho)$ are given by $s_0(\rho) = 1$, $s_1(\rho) = \sum_{i=1}^N \rho_i$, $s_2(\rho) = \sum_{i < j} \rho_i \rho_j$, etc. This defines the notation s_k . Then Eq. (9.99) is written as

$$\left[x \sum_{k=0}^{n_c} (-1)^k s_k(\phi^2) x^{n_c-k} + 2\Lambda m_1 \dots m_{n_f} \right]^2 - 4\Lambda^2 \prod_{i=1}^{n_f} (x + m_i^2) = -4\Lambda^2 x \prod_{a=1}^{n_c} (x - \alpha_a)^2. \quad (9.101)$$

Note that $s_k(\phi^2) = O(m^{2k-1}\Lambda)$ for $k \neq 0$ because one of the ϕ^2 's is $O(m\Lambda)$. This allows us to neglect $s_0(\phi^2)x^{n_c} = x^{n_c}$ term in the sum, and the equation becomes

$$\left[\sum_{k=1}^{n_c} (-1)^k s_k(\phi^2) x^{n_c-k+1} + 2\Lambda m_1 \dots m_{n_f} \right]^2 - 4\Lambda^2 \prod_{i=1}^{n_f} (x + m_i^2) = -4\Lambda^2 x \prod_{a=1}^{n_c} (x - \alpha_a)^2. \quad (9.102)$$

Now rewrite it as

$$\begin{aligned} & \prod_{i=1}^{n_f} (x + m_i^2) \\ &= x \left[\sum_{k=0}^{n_c} (-1)^k s_k(\alpha) x^{n_c-k} \right]^2 + \left[\sum_{k=1}^{n_c} (-1)^k \frac{s_k(\phi^2)}{2\Lambda} x^{n_c-k+1} + m_1 \dots m_{n_f} \right]^2 \\ &= x \left[\sum_{k=0}^{n_c} (-1)^k s_k(\alpha) x^{n_c-k} \right]^2 + \left[- \sum_{k=0}^{n_c-1} (-1)^k \frac{s_{k+1}(\phi^2)}{2\Lambda} x^{n_c-k} + m_1 \dots m_{n_f} \right]^2. \end{aligned} \quad (9.103)$$

Now consider the following polynomial

$$F(x) = \prod_{i=1}^{n_f} (\sqrt{x} + im_i) = \sum_{k=0}^{n_f} i^k s_k(m) \sqrt{x}^{n_f-k}. \quad (9.104)$$

This polynomial can be divided into the “real” and “imaginary” parts (this is not strictly true because m ’s are complex, but what is meant here is the division between terms of odd powers in i and of even powers in i),

$$\begin{aligned} F(x) &= \sum_{k=0}^{(n_f-1)/2} (-1)^k s_{2k}(m) \sqrt{x}^{n_f-2k} + \sum_{k=0}^{(n_f-1)/2} i(-1)^k s_{2k+1}(m) \sqrt{x}^{n_f-2k-1} \\ &= \sqrt{x} \sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} + i \sum_{k=0}^{n_c} (-1)^k s_{2k+1}(m) x^{n_c-k}, \end{aligned} \quad (9.105)$$

where we used $n_f = 2n_c + 1$. Similarly consider the polynomial

$$G(x) = \prod_{i=1}^{n_f} (\sqrt{x} - im_i) = \sqrt{x} \sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} - i \sum_{k=0}^{n_c} (-1)^k s_{2k+1}(m) x^{n_c-k}. \quad (9.106)$$

From the definition,

$$F(x)G(x) = \prod_{i=1}^{n_f} (x + m_i^2). \quad (9.107)$$

On the other hand, this product is also given by

$$F(x)G(x) = \left[\sqrt{x} \sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} \right]^2 + \left[\sum_{k=0}^{n_c} (-1)^{n_f-k} s_{2k+1}(m) x^k \right]^2. \quad (9.108)$$

This is precisely the same as Eq. (9.103) upon identifications

$$s_k(\alpha) = s_{2k}(m), \quad s_{k+1}(\phi^2) = -2\Lambda(-1)^{n_c} s_{2k+1}(m). \quad (9.109)$$

In the last identification, we used the fact that $s_{2n_c+1}(m) = s_{n_f}(m) = m_1 \cdots m_{n_f}$. This gives explicit solutions to the vacuum.

Once we have this solution, however, we can obtain other $2^{n_f-1} - 1$ solutions as follows. First note that the curve Eq. (6.24) is invariant under changing signs of even number of masses. Therefore, we can change signs of even number of masses from the solution (9.109). This gives 2^{n_f-1} solutions in total agreeing with \mathcal{N}_1 in the large μ analysis. This solution therefore decomposes under $U(n_f)$ for the equal mass case as $2^{n_f-1} = {}_{n_f}C_0 + {}_{n_f}C_2 + \cdots {}_{n_f}C_{n_f-1}$, reminiscent of the spinor representation.

The case of *equal mass* deserves further comments. As noted above, we can flip the signs of even number of quark masses, and therefore a general situation has $2r$ negative masses $-m$ and $n_f - 2r$ positive masses m . Let us study the location of branch points in this situation. The curve studied above has the form

$$y^2 = -4\Lambda^2 \prod_{a=1}^{n_c} (x - \alpha_a)^2 \quad (9.110)$$

near $x \sim 0$. Given the solutions Eq. (9.109), we can write

$$2\Lambda \prod_{a=1}^{n_c} (x - \alpha_a) = 2\Lambda \sum_{k=0}^{n_c} (-1)^k s_k(\alpha) x^{n_c-k}$$

$$\begin{aligned}
&= 2\Lambda \sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} \\
&= 2\Lambda \frac{1}{2\sqrt{x}} \left[\prod_{i=1}^{n_f} (\sqrt{x} + im_i) + \prod_{i=1}^{n_f} (\sqrt{x} - im_i) \right] \\
&= \Lambda \frac{1}{\sqrt{x}} \left[(\sqrt{x} + im_i)^{n_f-2r} (\sqrt{x} - im_i)^{2r} + (\sqrt{x} - im_i)^{n_f-2r} (\sqrt{x} + im_i)^{2r} \right].
\end{aligned} \tag{9.111}$$

Note that $1/\sqrt{x}$ does not introduce a singularity at $x = 0$. If $4r < n_f$, we can factor $(x + m^2)^{2r}$ from above and hence $(x + m^2)^{4r}$ from the curve. We interpret this factor as the emergence of $SU(2r)$ gauge theory. If $4r > n_f$, we can factor $(x + m^2)^{n_f-2r}$ from above and hence $(x + m^2)^{2n_f-4r}$ from the curve. We interpret this factor as the emergence of $SU(n_f - 2r) = SU(2(n_c - r) + 1)$ gauge theory. Combining both, we see that gauge groups up to $SU((n_f - 1)/2)$ are possible. This fact can also be understood from the Higgs branch picture. When quark masses are large and equal, one can cancel quark masses by the adjoint VEVs classically (squark singularity) and obtain $U(k)$ gauge theories $k = 0, 1, \dots, (n_f - 1)/2$ depending on how many components of the adjoint field is used to cancel the quark masses. These are all infrared-free theories and the gauge fields survive the quantum effects. While smoothly decreasing the quark masses, the Higgs branch emanating from the squark singularity does not change due to the non-renormalization theorem and hence the $U(k)$ theories survive to the small mass studied using the Coulomb branch. The effective low-energy Lagrangian which describes physics around the squark singularity is therefore nothing but that of Argyres–Plesser–Seiberg for $SU(n_c)$ theories at the non-baryonic roots [17]. We have shown (Section 8.1) that these theories indeed produce $n_f C_k$ vacua upon $\mu \neq 0$ and generic quark mass perturbation. Therefore the whole picture nicely fits together. On the other hand, the strictly massless case has the high singularity x^{2n_c} and appears to give a new superconformal theory with a global symmetry $SO(2n_f)$. We do not know of a weakly coupled description of this singularity in terms of a local field theory. We have checked that this singularity indeed produces mutually non-local dyons for $n_c = 2$ and $n_f = 5$.

Now we come back to the case of smaller n_f . Around the Chebyshev solution in the presence of quark masses, the curve is

$$\begin{aligned}
xy^2 &= \left[x \prod_{a=1}^{(n_f-1)/2} (x - \phi_a^2) \prod_{k=1}^{n_c-(n_f-1)/2} (x - \phi_k^2 - \kappa_k^2) + 2\Lambda^{2n_c+2-n_f} m_1 \cdots m_{n_f} \right]^2 \\
&\quad - 4\Lambda^{2(2n_c+2-n_f)} \prod_{i=1}^{n_f} (x + m_i^2),
\end{aligned} \tag{9.112}$$

where ϕ_k^2 are given by those for the Chebyshev polynomial. However, for the purpose of studying the behavior around $x = 0$, both x and the shifts κ_k^2 can be ignored relative to ϕ_k^2 .

We need to know $\prod_{k=1}^{n_c-(n_f-1)/2} (-\phi_k^2)$. This can be calculated as

$$\begin{aligned}
\prod_{k=1}^{n_c-(n_f-1)/2} (-\phi_k^2) &= \frac{2\Lambda^{2n_c+1-n_f}}{\sqrt{x}} T_{2n_c+2-n_f} \left(\frac{\sqrt{x}}{2\Lambda} \right) \Big|_{x \rightarrow 0} \\
&= \Lambda^{2n_c+1-n_f} \lim_{t \rightarrow 0} \frac{1}{t} \cos((2n_c+2-n_f) \arccos t) \\
&= (-1)^{n_c-(n_f-1)/2} (2n_c+2-n_f) \Lambda^{2n_c+1-n_f}.
\end{aligned} \tag{9.113}$$

Then the curve is approximated as

$$\begin{aligned}
\frac{xy^2}{\Lambda^{2(2n_c+1-n_f)}} &= \left[(-1)^{n_c-(n_f-1)/2} (2n_c+2-n_f) x \prod_{a=1}^{(n_f-1)/2} (x - \phi_a^2) + 2\Lambda m_1 \cdots m_{n_f} \right]^2 \\
&\quad - 4\Lambda^2 \prod_{i=1}^{n_f} (x + m_i^2).
\end{aligned} \tag{9.114}$$

This is nothing but the curve of $USp(2n'_c)$ theory with $n'_c = (n_f - 1)/2$ upon changing normalizations of x , y , ϕ_a^2 . The rest of the analysis therefore follows exactly the same as in $2n_c = n_f - 1$ case. Even when other Chebyshev solutions obtained by $Z_{2n_c+2-n_f}$ are used, they simply amount to the change of phase of Λ in the above approximate curve and the analysis remains the same.

ii) Chebyshev point: even n_f

Consider now even n_f cases. Again, let us study the specific case of $n_f = 2n_c$ first. We shall come back to the more general cases later on.

The Chebyshev solution is obtained in the massless limit by setting all but one of $\phi_a = 0$:

$$xy^2 = [x^{n_c}(x - \phi_{n_c}^2)]^2 - 4\Lambda^4 x^{2n_c} = x^{2n_c}(x - \phi_{n_c}^2 - 2\Lambda^2)(x - \phi_{n_c}^2 + 2\Lambda^2). \tag{9.115}$$

We take $\phi_{n_c}^2 = \pm 2\Lambda^2$ so that the zero at $x = 0$ has degree $2n_c$. There is another isolated zero at $x = \pm 4\Lambda^2$. There is also a branch point at $x = \infty$. We first consider the case $\phi_{n_c}^2 = +2\Lambda^2$ and will come back to the case $\phi_{n_c}^2 = -2\Lambda^2$ later on.

Under the perturbation by generic quark masses, we go back to the original curve. The only way that the curve

$$xy^2 = \left[x \prod_{a=1}^{n_c-1} (x - \phi_a^2)(x - 2\Lambda^2 - \beta) + 2\Lambda^2 m_1 \cdots m_{n_f} \right]^2 - 4\Lambda^4 \prod_{i=1}^{n_f} (x + m_i^2) \tag{9.116}$$

can be arranged to have n_c double zeros as

$$xy^2 = x(x - 4\Lambda^2 - \gamma) \prod_{a=1}^{n_c} (x - \alpha_a)^2, \tag{9.117}$$

is by assuming

$$\phi_a \sim m^2, \quad \alpha_{n_c} \sim m\Lambda, \quad \alpha_1^2 \sim \cdots \sim \alpha_{n_c-1}^2 \sim m^2. \tag{9.118}$$

By neglecting $\beta, \gamma, x \ll \Lambda^2$, we must solve the equation

$$\begin{aligned} & \left[-2\Lambda^2 x \prod_{a=1}^{n_c-1} (x - \phi_a^2) + 2\Lambda^2 m_1 \cdots m_{n_f} \right]^2 - 4\Lambda^4 \prod_{i=1}^{n_f} (x + m_i^2) \\ &= -4\Lambda^2 x \prod_{a=1}^{n_c} (x - \alpha_a)^2 = 4\Lambda^2 \alpha_{n_c}^2 x \prod_{a=1}^{n_c-1} (x - \alpha_a)^2. \end{aligned} \quad (9.119)$$

It is interesting to note that this is the curve for the superconformal $USp(n_f - 2)$ theory with n_f flavors with a special choice of $g = -1$. This can be rewritten as

$$\left[-x \sum_{k=0}^{n_c-1} (-1)^k s_k(\phi^2) x^{n_c-1-k} + m_1 \cdots m_{n_f} \right]^2 - \prod_{i=1}^{n_f} (x + m_i^2) = -\frac{\alpha_{n_c}^2}{\Lambda^2} x \prod_{a=1}^{n_c-1} (x - \alpha_a)^2. \quad (9.120)$$

By moving terms, we find

$$\begin{aligned} & \prod_{i=1}^{n_f} (x + m_i^2) \\ &= \left[-\sum_{k=0}^{n_c-1} (-1)^k s_k(\phi^2) x^{n_c-k} + m_1 \cdots m_{n_f} \right]^2 + \frac{\alpha_{n_c}^2}{\Lambda^2} x \left[\sum_{k=0}^{n_c-1} (-1)^k s_k(\alpha) x^{n_c-1-k} \right]^2. \end{aligned} \quad (9.121)$$

Now consider the following polynomial

$$F(x) = \prod_{i=1}^{n_f} (\sqrt{x} + i m_i) = \sum_{k=0}^{n_f} i^k s_k(m) \sqrt{x}^{n_f-k}. \quad (9.122)$$

This polynomial can be divided into the “real” and “imaginary” parts (this is not strictly true because m ’s are complex, but what is meant here is the division between terms of odd powers in i and of even powers in i),

$$\begin{aligned} F(x) &= \sum_{k=0}^{n_f/2} (-1)^k s_{2k}(m) \sqrt{x}^{n_f-2k} + \sum_{k=0}^{n_f/2-1} i (-1)^k s_{2k+1}(m) \sqrt{x}^{n_f-2k-1} \\ &= \sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} + i \sqrt{x} \sum_{k=0}^{n_c-1} (-1)^k s_{2k+1}(m) x^{n_c-k-1}, \end{aligned} \quad (9.123)$$

where we used $n_f = 2n_c$. Similarly consider the polynomial

$$G(x) = \prod_{i=1}^{n_f} (\sqrt{x} - i m_i) = \sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} - i \sqrt{x} \sum_{k=0}^{n_c-1} (-1)^k s_{2k+1}(m) x^{n_c-k-1}. \quad (9.124)$$

From the definition,

$$F(x)G(x) = \prod_{i=1}^{n_f} (x + m_i^2). \quad (9.125)$$

On the other hand, this product is also given by

$$F(x)G(x) = \left[\sum_{k=0}^{n_c} (-1)^k s_{2k}(m) x^{n_c-k} \right]^2 + \left[\sqrt{x} \sum_{k=0}^{n_c-1} (-1)^k s_{2k+1}(m) x^{n_c-k-1} \right]^2. \quad (9.126)$$

This is precisely the same as Eq. (9.121) upon identifications

$$\frac{1}{\Lambda} \alpha_{n_c} s_k(\alpha) = s_{2k+1}(m), \quad s_k(\phi^2) = -(-1)^{n_c} s_{2k}(m). \quad (9.127)$$

The first equation gives $\alpha_{n_c} = \Lambda s_1(m)$ by setting $k = 0$ and using $s_0 = 1$. In the last identification, we used the fact that $s_{2n_c}(m) = s_{n_f}(m) = m_1 \cdots m_{n_f}$. Note that $s_k(\phi^2)$ excludes $\phi_{n_c}^2$ and hence are different from the conventional gauge-invariant polynomials. This gives explicit solutions to the vacuum.

Once we have this solution, however, we can obtain other $2^{n_f-1} - 1$ solutions as follows. First note that the curve Eq. (6.24) is invariant under changing signs of even number of masses. Therefore, we can change signs of even number of masses from the solution (9.127). This gives 2^{n_f-1} solutions in total agreeing with \mathcal{N}_1 in the large μ analysis. This solution therefore decomposes under $U(n_f)$ for the equal mass case as $2^{n_f-1} = {}_{n_f}C_0 + {}_{n_f}C_2 + \cdots {}_{n_f}C_{n_f}$, *i.e.*, to even-rank anti-symmetric tensors, reminiscent of the spinor representation.

Now we come back to the other solution $\phi_{n_c}^2 = -2\Lambda^2$. This choice changes Eq. (9.121) to

$$\begin{aligned} & \prod_{i=1}^{n_f} (x + m_i^2) \\ &= \left[+ \sum_{k=0}^{n_c-1} (-1)^k s_k(\phi^2) x^{n_c-k} + m_1 \cdots m_{n_f} \right]^2 - \frac{\alpha_{n_c}^2}{\Lambda^2} x \left[\sum_{k=0}^{n_c-1} (-1)^k s_k(\alpha) x^{n_c-1-k} \right]^2. \end{aligned} \quad (9.128)$$

This is again the curve for the superconformal $USp(2n_f - 2)$ theory with n_f flavors with a different choice of $g = +1$. The sign changes can be absorbed if we flip the sign of m_1 and change α_{n_c} to $i\alpha_{n_c}$. The sign flip of one of the quark masses implies that we have an odd number of minus signs in quark masses. This solution therefore decomposes under $U(n_f)$ for the equal mass case as $2^{n_f-1} = {}_{n_f}C_1 + {}_{n_f}C_3 + \cdots {}_{n_f}C_{n_f-1}$, *i.e.*, to odd-rank anti-symmetric tensors, reminiscent of the anti-spinor representation.

Comments on the case of *equal mass* are in order. As noted above, we can flip the signs of even number of quark masses, and therefore a general situation has $2r$ negative masses $-m$ and $n_f - 2r$ positive masses m . Let us study the location of branch points in this situation. The curve studied above has the form

$$y^2 = -4\Lambda^2 \alpha_{n_c}^2 \prod_{a=1}^{n_c-1} (x - \alpha_a)^2 \quad (9.129)$$

near $x \sim 0$. Given the solutions Eq. (9.127), we can write

$$2\Lambda \alpha_{n_c} \prod_{a=1}^{n_c-1} (x - \alpha_a) = 2\Lambda \alpha_{n_c} \sum_{k=0}^{n_c-1} (-1)^k s_k(\alpha) x^{n_c-1-k} = 2\Lambda^2 \sum_{k=0}^{n_c-1} (-1)^k s_{2k+1}(m) x^{n_c-1-k}$$

$$\begin{aligned}
&= 2\Lambda^2 \frac{1}{2i} \sqrt{x} \left[\prod_{i=1}^{n_f} (\sqrt{x} + im_i) - \prod_{i=1}^{n_f} (\sqrt{x} - im_i) \right] \\
&= -i\Lambda^2 \sqrt{x} \left[(\sqrt{x} + im_i)^{n_f-2r} (\sqrt{x} - im_i)^{2r} - (\sqrt{x} - im_i)^{n_f-2r} (\sqrt{x} + im_i)^{2r} \right].
\end{aligned} \tag{9.130}$$

If $4r \leq n_f$, we can factor $(x + m^2)^{2r}$ from above and hence $(x + m^2)^{4r}$ from the curve. We interpret this factor as the emergence of $SU(2r)$ gauge theory. If $4r > n_f$, we can factor $(x + m^2)^{n_f-2r}$ from above and hence $(x + m^2)^{2n_f-4r}$ from the curve. We interpret this factor as the emergence of $SU(n_f - 2r) = SU(2(n_c - r))$ gauge theory. This fact can also be understood from the Higgs branch picture. When quark masses are large and equal, one can cancel quark masses by the adjoint VEV classically (squark singularity) and obtain $U(k)$ gauge theories $k = 0, 1, \dots, n_f/2 = n_c$ depending on how many components of the adjoint field is used to cancel the quark masses. These are all infrared-free or scale-invariant theories and the gauge fields survive the quantum effects. While smoothly decreasing the quark masses, the Higgs branch emanating from the squark singularity does not change due to the non-renormalization theorem and hence the $U(k)$ theories survive to the small mass studied using the Coulomb branch. The effective low-energy Lagrangian which describes physics around the squark singularity is therefore the one given by Argyres–Plesser–Seiberg for $SU(n_c)$ theories at the non-baryonic roots [17]. These theories indeed produce ${}_f C_k$ vacua upon $\mu \neq 0$ and generic quark mass perturbation. Therefore the whole picture nicely fit together. On the other hand, the strictly massless case has the high singularity x^{2n_c} and appears to give a new superconformal theory with a global symmetry $SO(2n_f)$. We do not know of a weakly coupled description of this singularity in terms of a weakly coupled local field theory. We have checked that this singularity indeed produces mutually non-local dyons for $n_c = 2$ and $n_f = 4$.

Now we come back to the case of smaller n_f . Around the Chebyshev solution in the presence of quark masses, we study the curve

$$\begin{aligned}
xy^2 &= \left[x \prod_{a=1}^{n_f/2-1} (x - \phi_a^2) \prod_{k=1}^{n_c+1-n_f/2} (x - \phi_k^2 - \kappa_k^2) + 2\Lambda^{2n_c+2-n_f} m_1 \cdots m_{n_f} \right]^2 \\
&\quad - 4\Lambda^{2(2n_c+2-n_f)} \prod_{i=1}^{n_f} (x + m_i^2),
\end{aligned} \tag{9.131}$$

where ϕ_k^2 are given by those for the Chebyshev polynomial. However, for the purpose of studying the behavior around $x = 0$, both x and the shifts κ_k^2 can be ignored relative to ϕ_k^2 . We need to know $\prod_{k=1}^{n_c+1-n_f/2} (-\phi_k^2)$. This can be calculated as

$$\begin{aligned}
\prod_{k=1}^{n_c+1-n_f/2} (-\phi_k^2) &= 2\Lambda^{2n_c+2-n_f} T_{2n_c+2-n_f} \left(\frac{\sqrt{x}}{2\Lambda} \right) \Big|_{x \rightarrow 0} \\
&= 2\Lambda^{2n_c+2-n_f} \cos \left[(2n_c + 2 - n_f) \arccos \frac{\sqrt{x}}{2\Lambda} \right] \Big|_{x \rightarrow 0} \\
&= 2(-1)^{n_c+1-n_f/2} \Lambda^{2n_c+2-n_f}.
\end{aligned} \tag{9.132}$$

Then the curve is approximated as

$$\frac{xy^2}{4\Lambda^{2(2n_c+2-n_f)}} = \left[(-1)^{n_c+1-n_f/2} x \prod_{a=1}^{n_f/2-1} (x - \phi_a^2) + m_1 \cdots m_{n_f} \right]^2 - \prod_{i=1}^{n_f} (x + m_i^2), \quad (9.133)$$

This is nothing but the left hand side of Eq.(9.119) for the effective curve of $USp(n_f)$ theory upon changing normalizations of x, y, ϕ_a^2 . The rest of the analysis therefore is exactly the same as in the $2n_c = n_f$ case. With other Chebyshev solutions obtained by $Z_{2n_c+2-n_f}$, the relative sign between the two terms in the square bracket changes. This sign change corresponds to two different solutions $\phi_{n_c}^2 = \pm 2\Lambda^2$ in the $2n_c = n_f$ case and hence they give decompositions into even-rank (odd-rank) anti-symmetric tensors under $U(n_f)$ for the equal mass perturbation, respectively.

iii) Special (baryonic-like) point

$USp(2n_c)$ theories also have special Higgs branch root (Eq.(6.30)-Eq.(6.32)), similar to the baryonic roots of the $SU(n_c)$ theories. This point is obtained by setting $\phi_1, \dots, \phi_{n_c-\tilde{n}_c} \neq 0$, and $\phi_{n_c-\tilde{n}_c+1} = \dots = \phi_{n_c} = 0$ in the original $USp(2n_c)$ curve

$$xy^2 = \left[x \prod_{a=1}^{n_c} (x - \phi_a^2) + \Lambda^{2n_c+2-n_f} \prod_{i=1}^{n_f} m_i \right]^2 - \Lambda^{4n_c+4-2n_f} \prod_{i=1}^{n_f} (x + m_i^2), \quad (9.134)$$

(here and below, $\tilde{n}_c = n_f - n_c - 2$), leading to

$$y^2 = x^{2\tilde{n}_c+1} \prod_{k=1}^{n_c-\tilde{n}_c} (x - \Phi_k)^2 - 4\Lambda^{4n_c+4-2n_f} x^{n_f-1}. \quad (9.135)$$

Nonvanishing Φ 's are taken as

$$(\Phi_1^2, \dots, \Phi_k^2, \dots, \Phi_{n_c-\tilde{n}_c}^2) = \Lambda^2(\omega, \dots, \omega^{2k-1}, \dots, \omega^{2(n_c-\tilde{n}_c)-1}), \quad (9.136)$$

where $\omega = e^{\pi i/(n_c-\tilde{n}_c)}$. Note that our ω is the square root of ω in Argyres–Plesser–Seiberg paper because of later convenience. Then the product $\prod_{k=1}^{n_c-\tilde{n}_c} (x - \Phi_k)$ can be rewritten as $x^{n_c-\tilde{n}_c} + \Lambda^{2(n_c-\tilde{n}_c)}$, and the curve becomes

$$\begin{aligned} y^2 &= x^{2\tilde{n}_c+1} \left[(x^{n_c-\tilde{n}_c} + \Lambda^{2(n_c-\tilde{n}_c)})^2 - 4\Lambda^{4n_c+4-2n_f} x^{n_c-\tilde{n}_c} \right] \\ &= x^{2\tilde{n}_c+1} (x^{n_c-\tilde{n}_c} - \Lambda^{2(n_c-\tilde{n}_c)})^2. \end{aligned} \quad (9.137)$$

The double zeros of the factor in the parenthesis are at

$$x = \Lambda^2 \omega^2, \Lambda^2 \omega^4, \dots, \Lambda^2 \omega^{2k}, \dots, \Lambda^2 \omega^{2(n_c-\tilde{n}_c)} = \Lambda^2. \quad (9.138)$$

One crucial difference from the $SU(n_c)$ case is that there is no choice but keep all of the “large” double zeros because the zeros at $x = 0$ has an odd power $2\tilde{n}_c + 1$ and hence leaves one of

the zeros not doubled anyway. This explains why the separation between \mathcal{N}_1 and \mathcal{N}_2 works out nicely with the $USp(2n_c)$ theories, but one of the r -roots gets mixed up with the baryonic root for $SU(n_c)$ theories. Another crucial difference is that there is no constraint among ϕ_a 's. Therefore we can just fix “large” ones and study the low-energy curve. Using

$$\prod_{k=1}^{n_c-\tilde{n}_c} (x - \Lambda^2 \omega^{2k-1}) = x^{n_c-\tilde{n}_c} + \Lambda^{2(n_c-\tilde{n}_c)} \simeq \Lambda^{2(n_c-\tilde{n}_c)}, \quad (9.139)$$

the curve is approximated as

$$\begin{aligned} xy^2 &= \left[x \prod_{a=1}^{\tilde{n}_c} (x - \phi_a^2) \Lambda^{2(n_c-\tilde{n}_c)} + \Lambda^{2n_c+2-n_f} \prod_{i=1}^{n_f} m_i \right]^2 - \Lambda^{4n_c+4-2n_f} \prod_{i=1}^{n_f} (x + m_i^2) \\ &= \Lambda^{4(n_c-\tilde{n}_c)} \left\{ \left[x \prod_{a=1}^{\tilde{n}_c} (x - \phi_a^2) + \frac{1}{\Lambda^{n_c-\tilde{n}_c}} \prod_{i=1}^{n_f} m_i \right]^2 - \frac{1}{\Lambda^{2(n_c-\tilde{n}_c)}} \prod_{i=1}^{n_f} (x + m_i^2) \right\}. \end{aligned} \quad (9.140)$$

This curve clearly describes an infrared free $USp(2\tilde{n}_c)$ theory.

Now we choose r out of \tilde{n}_c ϕ_a 's to match r of the mass-squared (${}_f C_r$ choices), *e.g.*,

$$\phi_1^2 = -m_1^2, \dots, \phi_r^2 = -m_r^2, \quad (9.141)$$

while the other ϕ_a^2 are still allowed to fluctuate but with magnitudes much less than m^2 . Note that the absence of constraints among ϕ_a allows us to have $r \leq \tilde{n}_c$ while we had $r < \tilde{n}_c$ in $SU(n_c)$ theories. Then the low-energy curve (9.140) can further be approximated as

$$xy^2 = \Lambda^{4(n_c-\tilde{n}_c)} \left[x \prod_{a=1}^{\tilde{n}_c-r} (x - \phi_a^2) \prod_{i=1}^r m_i^2 + \frac{1}{\Lambda^{n_c-\tilde{n}_c}} \prod_{i=1}^{n_f} m_i \right]^2 - \frac{1}{\Lambda^{2(n_c-\tilde{n}_c)}} \prod_{i=1}^{n_f} m_i^2, \quad (9.142)$$

which is nothing but the same as the curve of pure $USp(2(\tilde{n}_c - r + 1))$ Yang-Mills theories with the dynamical scale

$$\Lambda_{pure}^{2(\tilde{n}_c-r+1)} = \frac{1}{\Lambda^{n_c-\tilde{n}_c}} \frac{\prod_{i=1}^{n_f} m_i}{\prod_{i=1}^r m_i^2} \sim m^{2(\tilde{n}_c-r+1)} \left(\frac{m}{\Lambda} \right)^{n_c-\tilde{n}_c} \ll m^{2(\tilde{n}_c-r+1)}. \quad (9.143)$$

This justifies the assumption of $\phi_a^2 \ll m^2$. Since this is the curve of pure $USp(2(\tilde{n}_c - r + 1))$ Yang-Mills theories, it gives $(\tilde{n}_c - r + 1)$ vacua, and hence in total we find

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c} (\tilde{n}_c - r + 1) {}_f C_r. \quad (9.144)$$

iv) Summary of the vacuum counting in $USp(2n_c)$ theories

There are thus two groups of $N = 1$ vacua predicted by the Seiberg-Witten curve in $USp(2n_c)$ theories. The Chebyshev point Eq.(6.27), Eq.(6.28), gives rise to $\mathcal{N}_1 = (2n_c + 2 - n_f) \cdot 2^{n_f-1}$ vacua upon mass perturbation, while the special point Eq.(6.30), Eq.(6.31), leads to $\mathcal{N}_2 = \sum_{r=0}^{\tilde{n}_c} (\tilde{n}_c - r + 1) {}_f C_r$ vacua. Their sum coincides with the total number of $N = 1$ vacua found from the semiclassical as well as from large μ analyses.

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References

- [1] N. Seiberg and E. Witten, Nucl.Phys. **B426** (1994) 19; Erratum *ibid.* **B430** (1994) 485, hep-th/9407087.
- [2] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484, hep-th/9408099.
- [3] P. C. Argyres and A. F. Faraggi, Phys. Rev. Lett. **74** (1995) 3931, hep-th/9411047; A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. **B344** (1995) 169, hep-th/9411048; Int. J. Mod. Phys. A11 (1996) 1929-1974, hep-th/9505150; A. Hanany and Y. Oz, Nucl. Phys. **B452** (1995) 283, hep-th/9505075; P. C. Argyres, M. R. Plesser and A. D. Shapere, Phys. Rev. Lett. **75** (1995) 1699, hep-th/9505100; P. C. Argyres and A. D. Shapere, Nucl. Phys. **B461** (1996) 437, hep-th/9509175; A. Hanany, Nucl.Phys. **B466** (1996) 85, hep-th/9509176.
- [4] N. Seiberg, Nucl. Phys. **B435** (1995) 129, hep-th/9411149.
- [5] K. Intriligator and N. Seiberg, Nucl. Phys. **B444** (1995) 125, hep-th/9503179; K. Intriligator and P. Pouliot, Phys. Lett. **B353** (1995) 471, hep-th/9505006.
- [6] T. Kutasov, Phys. Lett. **B351** (1995) 230, hep-th/9503086; T. Kutasov and A. Schwimmer, *ibid.* **B354** (1995) 315, hep-th/9505004; T. Kutasov, A. Schwimmer and N. Seiberg Nucl.Phys. **B459** (1996) 455, hep-th/9510222.
- [7] P. C. Argyres and M.R. Douglas, Nucl. Phys. **B452** (1995) 283, hep-th/9505062.
- [8] P. C. Argyres, M. R. Plesser, N. Seiberg and E. Witten, Nucl. Phys. **461** (1996) 71, hep-th/9511154.
- [9] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, Nucl. Phys. **B471** (1996) 430, hep-th/9603002.
- [10] N. Evans, S. D. H. Hsu, M. Schwetz and S. B. Selipsky, Nucl. Phys. **B456** (1995) 205, hep-th/9508002.
- [11] M. Di Pierro and K. Konishi, Phys. Lett. **B388** (1996) 90, hep-th/9605178; K. Konishi, Phys. Lett. **B392** (1997) 101, hep-th/9609021.

- [12] L. Álvarez-Gaumé, M. Marino and F. Zamora, Int. J. Mod. Phys. **A13** (1998) 403, hep-th/9703072.
- [13] K. Konishi and H. Terao, Nucl. Phys. **B511** (1998) 264, hep-th/9707005; G. Carlino, K. Konishi and H. Terao, JHEP **04** (1998) 003, hep-th/9801027.
- [14] G. 't Hooft, Nucl. Phys. **B190** (1981) 455.
- [15] Y. Nambu, Phys. Rev. **D10** (1974) 4262, S. Mandelstam, Phys. Lett. **53B** (1975) 476; Phys. Rep. **23C** (1976) 245.
- [16] K. Konishi and K. Takenaga, hep-th/9911097, Phys. Lett B to appear.
- [17] P. C. Argyres, M. R. Plesser and N. Seiberg, Nucl. Phys. **B471** (1996) 159, hep-th/9603042.
- [18] P.C. Argyres, M.R. Plesser, and A.D. Shapere, Nucl. Phys. **B483** (1997) 172, hep-th/9608129.
- [19] R. Jackiw and C. Rebbi, Phys. Rev. **D13** (1976) 3398.
- [20] G. Carlino, K. Konishi and H. Murayama, JHEP **0002** (2000) 004, hep-th/0001036.
- [21] S. Coleman and E. Witten, Phys. Rev. Lett. **45** (1980) 100.
- [22] T. Hiramaya, N. Maekawa and S. Sugimoto, Progr. Theor. Phys. **99** (1998) 843, hep-th/9705069.
- [23] N. Seiberg, Phys. Rev. **D49** (1994) 6857, hep-th/9402044. K. Intriligator, R.G. Leigh and N. Seiberg, Phys. Rev. **D50** (1994) 1092, hep-th/9403198.
- [24] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, Phys. Rep. **C162** (1988) 169.
- [25] M.R. Douglas and S.H. Shenker, Nucl. Phys. **B447** (1995) 271, hep-th/9503163; A. Brandhuber and K. Landsteiner, Phys. Lett. **B358** (1995) 73, hep-th/9507008
- [26] J. A. Minahan and D. Nemeschansky, Nucl. Phys. **B464** (1996) 3, hep-th/9507032.1
- [27] R. Jackiw and C. Rebbi, Phys. Rev. Lett **36** (1976) 1116; P. Hasenfratz and G. 't Hooft, *ibid.* 1119; A. S. Goldharber, *ibid.* 1122.

Appendix A $SO(2N) \cap USp(2N) = U(N)$

The $SO(2N)$ generators are the most general pure imaginary anti-symmetric matrices. Break it down to the $N \times N$ blocks, and write them down as:

$$\begin{pmatrix} E & F \\ -{}^t F & D \end{pmatrix}, \quad (\text{A.1})$$

where D, E, F are all pure imaginary $N \times N$ matrices, with the constraints ${}^t E = -E$, ${}^t D = -D$. The generators of $USp(2N)$ are given by

$$\begin{pmatrix} B & A \\ C & -{}^t B \end{pmatrix}, \quad (\text{A.2})$$

with the constraints, ${}^t A = A$, ${}^t C = C$, $A^* = C$, $B^\dagger = B$.

The way to compare them is to go to the bases of $SO(2N)$ where it naturally breaks to $N + \bar{N}$ under $U(N)$. This can be done by the following rotation,

$$\begin{aligned} & \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} E & F \\ -{}^t F & D \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (E + D) + i(F + {}^t F) & i(E - D) + (F - {}^t F) \\ -i(E - D) + (F - {}^t F) & (E + D) - i(F + {}^t F) \end{pmatrix}. \end{aligned} \quad (\text{A.3})$$

Since both E, D are anti-symmetric, $(E + D)$ in the 1st block is the most general anti-symmetric imaginary matrix, while $i(F + {}^t F)$ is the most general symmetric real matrix. Their sum gives the most general hermitian matrix. Comparing to the $USp(N)$ generators, the off-diagonal blocks are completely symmetric for $USp(N)$ and completely anti-symmetric for $SO(2N)$, and hence there is no overlap. While the diagonal blocks are the most general $N \times N$ hermitian matrices, and overlap completely, hence $SO(2N) \cap USp(2N) = U(N)$.

Appendix B Semiclassical Monopole States

The Jackiw–Rebbi zero mode has the form

$$\psi_L^{(0)} = ib \eta(x); \quad \psi_R^{(0)} = b \eta(x). \quad (\text{B.1})$$

in the chiral representation, where the commutation relations are the standard one:

$$\{b^i, (b^j)^\dagger\} = \delta^{ij}. \quad (\text{B.2})$$

Given the monopole state $|\Omega\rangle$, one can construct 2^{n_f} positive-norm states by acting various number of creation operators upon it,

$$(b^{i_1})^\dagger (b^{i_2})^\dagger \dots (b^{i_k})^\dagger |\Omega\rangle, \quad (\text{B.3})$$

which are all spinless bosons [27]. In general $SU(n_c)$ theories ($n_c \geq 3$) with n_f flavors the Noether currents are:

$$J_\mu^A = \bar{\psi}_D^i \gamma_\mu \lambda_{ij}^A \psi_D^j \quad (\text{B.4})$$

so that the charge operators are

$$Q^A = (b^i)^\dagger \lambda_{ij}^A b_j + \text{non zero modes.} \quad (\text{B.5})$$

The semiclassical monopole multiplets are formed by the state $|\Omega\rangle$ which is a singlet, the n_f states $(b^i)^\dagger |\Omega\rangle$ belonging to a \underline{n}_f , the ${}_f C_2$ states $(b^i)^\dagger (b^j)^\dagger |\Omega\rangle$ which is the second rank antisymmetric tensor, etc. Although semi-classically 2^{n_f} states (B.3) are all degenerates, higher quantum effects lift such a degeneracy in general, and only those belonging to the same irreducible representation will have the same mass (monopole multiplets).

In the $USp(2n_c)$ gauge theories the fermionic part of the lagrangian is

$$\sum_{i=1}^{2n_f} \left[\bar{\psi}_a^i i \bar{\sigma}^\mu (\mathcal{D}_\mu)_{ab} \psi_b^i + \frac{1}{\sqrt{2}} \psi_a^i \phi_{ab} \psi_c^i J^{bc} + \text{h.c.} \right], \quad (\text{B.6})$$

where all fermions are pure left-handed. In this basis of fermions the $SO(2n_f)$ symmetry is manifest and the global symmetry current is simply

$$J_\mu^{ij} = \bar{\psi}_a^i \bar{\sigma}^\mu \psi_a^j - (i \rightarrow j), \quad (\text{B.7})$$

their charges ($SO(2n_f)$ generators) are

$$Q_\mu^{ij} = \bar{\psi}_a^i \psi_a^j - (i \rightarrow j). \quad (\text{B.8})$$

The zero mode operators in $USp(2n_c)$ theories have the form (γ^i are just the name of these operators, not gamma matrices)

$$\psi^i = \gamma^i \eta(x) + \dots, \quad (i = 1, 2, \dots, 2n_f), \quad (\text{B.9})$$

where

$$\gamma^i = b^i + b^{i\dagger}; \quad \gamma^{n_f+i} = \frac{1}{i} (b^i - b^{i\dagger}), \quad (i = 1, 2, \dots, n_f). \quad (\text{B.10})$$

This particular form of the zero mode contribution reflects the fact that the fermion basis in which the standard Jackiw–Rebbi solution Eq. (B.1) applies and the one which transforms as an $SO(2n_f)$ vector, are related by:

$$\psi_a^i = \frac{1}{\sqrt{2}} (\hat{\psi}_a^{2i-1} + \hat{\psi}_a^{2i}), \quad \psi_a^{n_f+i} = \frac{1}{i\sqrt{2}} (\hat{\psi}_a^{2i-1} - \hat{\psi}_a^{2i}), \quad (i = 1, 2, \dots, n_f). \quad (\text{B.11})$$

where

$$\hat{\psi}_a^{2i-1} \equiv \psi_{La}^i \quad \hat{\psi}_a^{2i} \equiv \bar{\psi}_{Ra}^i, \quad (i = 1, \dots, n_f). \quad (\text{B.12})$$

Note that γ^i 's are all real. These relations show that γ^i 's obey the Clifford algebra,

$$\{\gamma^i, \gamma^j\} = 2\delta_{ij}. \quad (\text{B.13})$$

By substituting Eq. (B.9) into Eq. (B.8) we find that the generators of $SO(2n_f)$ symmetry are (by renormalizing by a constant):

$$\Sigma_{ij} = \frac{1}{4i} [\gamma^i, \gamma^j], \quad (\text{B.14})$$

which obviously satisfies the standard $SO(2n_f)$ algebra.

This shows that various monopole states:

$$(b^{i_1})^\dagger (b^{i_2})^\dagger \dots (b^{i_k})^\dagger |\Omega\rangle. \quad (\text{B.15})$$

transform as spinor representations of $SO(2n_f)$. Furthermore one notes that states with odd or even numbers of creation operators have definite “chirality” with respect to

$$\gamma^{2N_f+1} = (-i)^{N_f} \gamma^1 \gamma^2 \dots \gamma^{2N_f} = \prod_{i=1}^{N_f} (1 - 2b^{i\dagger} b^i) \quad (\text{B.16})$$

so that each of them transform independently. Each monopole state thus belongs to a spinor representation of definite chirality of the global $SO(2n_f)$ group.

Appendix C Explicit Formulae for a_{Di} , a_i , $\frac{\partial a_{Di}}{\partial u_j}$, and $\frac{\partial a_i}{\partial u_j}$

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda^{2n_c - n_f} \prod_{j=1}^{n_f} (x + m_j), \quad SU(n_c), \quad n_f \leq 2n_c - 2, \quad (\text{C.1})$$

and

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda \prod_{j=1}^{n_f} \left(x + m_j + \frac{\Lambda}{n_c} \right), \quad SU(n_c), \quad n_f = 2n_c - 1, \quad (\text{C.2})$$

with ϕ_k subject to the constraint $\sum_{k=1}^{n_c} \phi_k = 0$, and

$$xy^2 = \left[x \prod_{a=1}^{n_c} (x - \phi_a^2)^2 + 2\Lambda^{2n_c+2-n_f} m_1 \dots m_{n_f} \right]^2 - 4\Lambda^{2(2n_c+2-n_f)} \prod_{i=1}^{n_f} (x + m_i^2), \quad USp(2n_c). \quad (\text{C.3})$$

In each case, these represent a genus $g = n_c - 1$ ($g = n_c$ for the $USp(2n_c)$ case) hypertorus, which are characterized by $2g$ homology cycles $\alpha_i, \beta_i, i = 1, 2, \dots, g$. These cycles are taken in the doubly sheeted x - plane to surround two branch points of y , and such that they intersect pairwise, in the canonical way, $(\alpha_i \cdot \beta_j) = \delta_{ij}$. $\partial a_{Di}/\partial u_j$, and $\partial a_i/\partial u_j$ are given by the $g \times 2g$ period integrals of the holomorphic differentials (neglecting the normalization constant) [1]-[3]

$$\frac{\partial a_{Di}}{\partial u_j} = \oint_{\alpha_i} \frac{dx x^{j-1}}{y}; \quad \frac{\partial a_i}{\partial u_j} = \oint_{\beta_i} \frac{dx x^{j-1}}{y}; \quad (\text{C.4})$$

whereas a_{Di}, a_i are given by the integrals of the meromorphic differential λ (defined such as $d\lambda/du_j = \frac{dx x^{j-1}}{y}$;

$$a_{Di} = \oint_{\alpha_i} \lambda, \quad a_i = \oint_{\beta_i} \lambda, \quad (\text{C.5})$$

where some additive terms proportional to the bare quark masses are neglected.

Appendix D Absence of the Non-Baryonic Root with $r = \tilde{n}_c = n_f - n_c$

In this Appendix we prove the absence of the non-baryonic root with $r = \tilde{n}_c = n_f - n_c$ for $SU(n_c)$ gauge theory. The nonbaryonic branch root in question is characterized by the adjoint VEVs

$$\text{diag } \phi = (\underbrace{0, 0, \dots, 0}_{n_f - n_c}, \phi_1, \dots, \phi_{2n_c - n_f}), \quad \sum_{a=1}^{2n_c - n_f} \phi_a = 0 : \quad (\text{D.1})$$

the curve has the form

$$\begin{aligned} y^2 &= x^{2\tilde{n}_c} \prod_{a=1}^{2n_c - n_f} (x - \phi_a)^2 - 4\Lambda^{2n_c - n_f} x^{n_f} \\ &= x^{2\tilde{n}_c} \left[\prod_{a=1}^N (x - \phi_a)^2 - 4\Lambda^N x^N \right], \quad N = 2n_c - n_f. \end{aligned} \quad (\text{D.2})$$

We prove below that this curve cannot be put (for whatever $\{\phi_a\}$) in the form,

$$y^2 = x^{2\tilde{n}_c} \prod_{i=1}^{N-1} (x - \alpha_i)^2 (x - \gamma)(x - \delta), \quad \gamma \neq \delta. \quad (\text{D.3})$$

1. *Theorem:*

The function

$$F(x) = \prod_{a=1}^N (x - \phi_a)^2 - 4x^N, \quad \prod_{a=1}^N \phi_a \neq 0, \quad (\text{D.4})$$

$x, \{\phi\}$ complex, cannot have exactly $N - 1$ double factors.

2. $N = 2, 3, 4$

For $N = 2, 3, 4$, we have checked explicitly that there are indeed no ϕ configurations such that $F(x)$ has exactly $N - 1$ pairs of double factors. There are either N pairs, as can be realized by taking

$$\phi_a = (\omega_0)^a, \quad \omega_0 = e^{2\pi i/N}, \quad (\text{D.5})$$

or less than $N - 1$ pairs of double factors.

3. $N = 2n_c - n_f$ even

In this case,

$$F(x) = F_+(x)F_-(x), \quad F_{\pm}(x) = \prod_{a=1}^N (x - \phi_a) \pm 2x^{N/2}. \quad (\text{D.6})$$

First of all, there cannot be any common factor between $F_+(x)$ and $F_-(x)$. For if there is one, $(x - x^*)$, $F_+(x^*) = F_-(x^*) = 0$, hence $x^* = 0$. It means that there is an extra power of x^2 in front (an extra $\phi_a = 0$), which is not possible because $\prod_{a=1}^N \phi_a \neq 0$.

Since there are no common factor in $F_+(x)$ and $F_-(x)$, in order to get at least $N - 1$ double factors, one of $F_+(x)$ and $F_-(x)$ must be fully doubled up, say:

$$F_+(x) = \prod_{a=1}^N (x - \phi_a) + 2x^{N/2} = \prod_{a=1}^{N/2} (x - \alpha_a)^2. \quad (\text{D.7})$$

We wish to prove that in this case $F_-(x)$ is a perfect square also.

In order to show it, note that $F(x)$ is invariant under the transformation,

$$x \rightarrow \omega x; \quad \phi_a \rightarrow \omega \phi_a, \quad (\text{D.8})$$

where $\omega = \exp 2\pi i/N$. Note that under this transformation, $F_+(x)$ and $F_-(x)$ get exchanged:

$$F_+(x) \rightarrow F_-(x); \quad F_-(x) \rightarrow F_+(x). \quad (\text{D.9})$$

Assume now that a configuration $\{\phi\}$ such that (D.7) holds was found. α_i 's are functions of $\{\phi\}$:

$$\alpha_i = \alpha_i(\{\phi\}). \quad (\text{D.10})$$

According to (D.9), $F_-(x)$ can be found by the ω transformation from $F_+(x)$:

$$F_-(x) = \prod_{i=1}^{N/2} [x - \omega^{-1} \alpha_i(\{\omega\phi\})]^2. \quad (\text{D.11})$$

Thus we have proved that if $F(x)$ has at least $N - 1$ double factors, then it has N of them.

4. General $N = 2n_c - n_f$

Assume that $\{\phi_a\}$'s are found such that

$$F(x) = \prod_{a=1}^N (x - \phi_a)^2 - 4x^N \equiv \prod_{A=1}^{N-1} (x - \alpha_A)^2 (x - \gamma)(x - \delta), \quad \gamma \neq \delta, \quad (\text{D.12})$$

where α_A 's are all different among each other and none of them coincides either with γ or with δ . The left hand side of Eq.(D.12) is invariant under the transformation Eq.(D.8), so must be also the right hand side. That is

$$\begin{aligned} & \prod_{A=1}^{N-1} (x - \omega^{-1} \alpha_A(\{\omega\phi\}))^2 (x - \omega^{-1} \gamma(\{\omega\phi\}))(x - \omega^{-1} \delta(\{\omega\phi\})) \\ &= \prod_{A=1}^{N-1} (x - \alpha_A)^2 (x - \gamma)(x - \delta); \end{aligned} \quad (\text{D.13})$$

it implies however

$$\prod_{A=1}^{N-1} (x - \omega^{-1} \alpha_A(\{\omega\phi\})) = \prod_{A=1}^{N-1} (x - \alpha_A), \quad (\text{D.14})$$

$$(x - \omega^{-1} \gamma(\{\omega\phi\}))(x - \omega^{-1} \delta(\{\omega\phi\})) = (x - \gamma)(x - \delta). \quad (\text{D.15})$$

Now Eq.(D.14) and Eq.(D.15), which are equivalent to

$$\prod_{A=1}^{N-1} (\omega x - \alpha_A(\{\omega\phi\})) = \omega^{N-1} \prod_{A=1}^{N-1} (x - \alpha_A(\{\phi\})), \quad (\text{D.16})$$

$$(\omega x - \gamma(\{\omega\phi\}))(\omega x - \delta(\{\omega\phi\})) = \omega^2 (x - \gamma(\{\phi\}))(x - \delta(\{\phi\})), \quad (\text{D.17})$$

imply that the polynomials

$$H_1(x, \phi) = \prod_{A=1}^{N-1} (x - \alpha_A), \quad (\text{D.18})$$

$$H_2(x, \phi) = (x - \gamma)(x - \delta), \quad (\text{D.19})$$

of order $N - 1$ and 2, are both homogeneous in $(\{\phi\}, x)$: namely,

$$H_1(x, \phi) = x^{N-1} + \sum_{i=1}^{N-1} s_i(\phi) x^{N-1-i}, \quad (\text{D.20})$$

$$H_2(x, \phi) = x^2 + \sum_{i=1}^2 t_i(\phi) x^{2-i}, \quad (\text{D.21})$$

where $s_i(\phi)$ and $t_i(\phi)$ satisfy

$$s_i(\omega\phi) = \omega^i s_i(\phi); \quad t_i(\omega\phi) = \omega^i t_i(\phi). \quad (\text{D.22})$$

This is so because H_1, H_2 , being polynomials in x of order less than N , have each term in them transformed non trivially under the ω transformation.

It follows now that

$$F(x) = H_1(x)^2 \cdot H_2(x) \quad (\text{D.23})$$

is a homogeneous expression in $(\{\phi\}, x)$ with nontrivial coefficients in the expansion in x , which contradicts the form of $F(x)$, Eq.(D.4). We have thus shown that the assumption Eq.(D.12) is impossible.

Appendix E Monodromies in $SU(3)$ Theories with $n_f = 4$

In this Appendix we briefly describe the analysis of monodromy transformation around various singularities for $SU(3)$ gauge theory with $n_f = 4$. To study the monodromies one sets (u, v) slightly off the singularity interested, and lets (u, v) make a small circle around it in the parameter space (QMS). From the way the branch points move around and the branch cuts get entangled one easily finds the monodromy matrix for a_{D1}, a_{D2}, a_1, a_2 . One must also study how the positions of the branch points and cuts are varied as one goes from one singularity to another. This allows one to determine the homology cycles defining various periods, a_{D1}, a_{D2}, a_1, a_2 , in a globally consistent manner. The quantum numbers of the massless states at each singularity are found from the non-vanishing eigenvectors of the monodromy matrix thus obtained. In many cases, the movements of the branch points can be studied analytically as well: we illustrate below how such a check can be made, in some examples.

The $SU(3)$ gauge theory with four flavor has 17 vacua for generic quark masses and for a nonvanishing adjoint mass. They collapse to six vacua in the limit of equal quark masses, three singlets, two quartets and one sextet. For $\Lambda = 2$ and $m = 2^{-6}$ they are at:

1.	$(u, v) = (0, 0)$	sextet
2.	$(u, v) = (-0.92, -0.14)$	singlet
3.	$(u, v) = (-0.85, 0.09)$	singlet
4.	$(u, v) = (-1.05, -0.02)$	quartet
5.	$(u, v) = (-0.95, -0.01)$	quartet
6.	$(u, v) = (-1.00, -0.06)$	singlet

The branch points and cuts (dotted lines) are chosen as shown in Fig. 9. Let us analyze each singularity, starting from the singularity 2.

Singularity 2.

The branch points near this singularity are located as in (1.1) in Fig. 10 with $x_2 \equiv x_3$ and $x_5 \equiv x_6$ on the singularity.

To determine the massless BPS states condensing on the singularity, we perform a small circle around the singularity itself in the parameter space. The branch points transform as in 1.2 (Fig. 10) therefore the monodromy matrix is:

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \quad (\text{E.1})$$

The eigenvectors, with unimodular eigenvalues, of the (transpose of the) monodromy matrix give

the charges of the massless particles condensing in the singularity. In this case, we have:

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (1, 0, 0, 0), (0, 1, 0, 0), \quad (\text{E.2})$$

i.e., the two monopoles of the two abelian factors.

Singularity 3.

The x -plane is shown in 1.3 of Fig. 10. On the singularity: $x_1 \equiv x_2$ and $x_4 \equiv x_5$. The monodromy matrix is (see 1.4 Fig. 10):

$$M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}. \quad (\text{E.3})$$

The charges of the condensing particles are:

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (0, 1, 0, 1), (1, 0, 1, 0), \quad (\text{E.4})$$

i.e. two dyons of the two abelian factors. By an appropriate redefinition of $(n_{m1}, n_{m2}, n_{e1}, n_{e2})$ they become

$$(n'_{m1}, n'_{m2}, n_{e1}, n_{e2}) = (0, 1, 0, 0), (1, 0, 0, 0) : \quad (\text{E.5})$$

two monopoles of $U(1)^2$.

Singularity 6.

At the singularity 6 the coalescing branch points are: $x_1 \equiv x_2$ and $x_5 \equiv x_6$. See (1.5), (1.6) of Figs. 11. The monodromy matrix is:

$$M_6 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \quad (\text{E.6})$$

The charges of the condensing states:

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (1, 0, 1, 0), (0, 1, 0, 0). \quad (\text{E.7})$$

Note that, these objects are mutually local. By an appropriate redefinition of $(n_{m1}, n_{m2}, n_{e1}, n_{e2})$ they become again

$$(n''_{m1}, n_{m2}, n_{e1}, n_{e2}) = (0, 1, 0, 0), (1, 0, 0, 0) : \quad (\text{E.8})$$

two monopoles of $U(1)^2$.

Singularity 1.

This is the sextet singularity. The branch points (x -plane) are in the positions depicted in the 1.7 (Fig. 11), with $x_2 \equiv x_3 \equiv x_4 \equiv x_5$ exactly on the singularity. Performing a small circle around the

singularity (in the QMS) the branch points move as indicated in Fig. 1.8. We find the monodromy matrix

$$M_1 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ -2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{pmatrix}. \quad (\text{E.9})$$

so the massless particles have the quantum numbers:

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (0, 1, 0, 1), (1, 2, 2, 0), (1, 1, 0, 1). \quad (\text{E.10})$$

The first and the second of these states are relatively nonlocal, hence this is a conformally invariant vacuum. This was to be expected since this singularity corresponds to a class 3 conformal theory of Eguch et. al. (see the main text).

Singularity 4, 5.

For singularity 4, the x -plane looks like in 1.9 (Fig. 12) and the branch points rotates as in Fig. 1.10. The coincident branch points, on the singularity are: $x_3 \equiv x_4$ and $x_5 \equiv x_6$.

As for the singularity 5, the x -plane looks like in 1.11 and the branch points rotates as in 1.12. The coincident branch points, on the singularity are: $x_1 \equiv x_2$ and $x_3 \equiv x_4$.

The monodromy matrix at the singularity 4 turns out to be

$$M_4 = \begin{pmatrix} -3 & -4 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 4 & 5 & 0 \\ 4 & 5 & 4 & 1 \end{pmatrix}. \quad (\text{E.11})$$

so the massless particles have the quantum numbers:

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (1, 0, 1, 0), (0, 1, 0, 0). \quad (\text{E.12})$$

They are relatively local. Note that

$$M_4 = T^4 A, \quad T = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}, \quad (\text{E.13})$$

with the matrix T having the same charge eigenvectors (E.12) and A representing a possible change of homology cycles. This shows that the this singularity correspond to a quartet of singularity.

The monodromy matrix at the singularity 5 is

$$M_5 = \begin{pmatrix} -4 & -4 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 4 & 6 & 0 \\ 4 & 5 & 4 & 1 \end{pmatrix}. \quad (\text{E.14})$$

so the massless particles have the same quantum numbers as at 4:

$$(n_{m1}, n_{m2}, n_{e1}, n_{e2}) = (1, 0, 1, 0), (0, 1, 0, 0). \quad (\text{E.15})$$

They are again relatively local and the same as at the point 4.

Analytic determination of the monodromy

One can actually study the movement of the branch points analytically in many cases. For instance take the singularities 4 or 5 and consider the double branch point at $-m = -\frac{1}{64}$. (At both singularities, one of the double branch point occurs at $x = -m$.) The auxiliary curve

$$y^2 = -4 \left(x + \frac{1}{64} \right)^4 + (-v - ux + x^3)^2, \quad (\text{E.16})$$

can be rewritten as:

$$y^2 = -4 \left(x + \frac{1}{64} \right)^4 + \left\{ \left(x + \frac{1}{64} \right) f(x) - (u - u_0)x - (v - v_0) \right\}^2 \quad (\text{E.17})$$

with

$$f(x) = x^2 - \frac{1}{64}x - 64v_0. \quad (\text{E.18})$$

The positions of the singularities 4 and 5 are given by

$$u_0 = -\frac{17149}{16384}; \quad v_0 = \frac{u_0}{64} = -\frac{17149}{1048576} \quad (\text{E.19})$$

(the vacuum 4), and by

$$u_0 = -\frac{15613}{16384}; \quad v_0 = \frac{u_0}{64} = -\frac{15613}{1048576} \quad (\text{E.20})$$

(the vacuum 5).

Now, shift x by

$$x + \frac{1}{64} \equiv x' \quad (\text{E.21})$$

and rewrite the curve as:

$$y^2 = -4x'^4 + \left\{ x' f \left(x' - \frac{1}{64} \right) - (u - u_0) \left(x' - \frac{1}{64} \right) - (v - v_0) \right\}^2. \quad (\text{E.22})$$

For small but nonzero values of $u - u_0$ and/or $v - v_0$ of order ϵ , the second term of the right hand side has the form

$$\{(c_0 x' + c_1 x'^2 + \dots) + \epsilon x' + \epsilon\}^2. \quad (\text{E.23})$$

and the curve looks like

$$y^2 \simeq x'^4 + (x' + \epsilon)^2. \quad (\text{E.24})$$

Shifting further x' as $\tilde{x} = x' + \epsilon$ one gets

$$y^2 \simeq \tilde{x}^2 + (\tilde{x} - \epsilon)^4. \quad (\text{E.25})$$

The approximately doublet zeros of the right hand side are then found from

$$\tilde{x}^2 + \epsilon^4 - 4\epsilon^3\tilde{x} + 6\epsilon^2\tilde{x}^2 - 4\epsilon\tilde{x}^3 + \tilde{x}^4 = 0 \quad (\text{E.26})$$

to be of order $\tilde{x} \sim \pm\epsilon^2$. The splitting of the double branch point $-\frac{1}{64}$ is therefore given by:

$$x' \sim \epsilon \pm \epsilon^2, \quad \therefore \quad x \sim -\frac{1}{64} + \epsilon \pm \epsilon^2. \quad (\text{E.27})$$

A small circular motion in QMS around u_0 and/or v_0 yields a convoluted circular movements of the split double zeros: their relative position makes a 4π rotation (this is relevant to the monodromy analysis) while their center of mass performs a single 2π rotation.

This movement of the branch points has been confirmed by the numerical analysis (Fig. 1.10.).

Now, consider the other double zero at $x_0 \simeq 1$ (in the case of the singularity No. 4). The curve Eq.(E.17) becomes

$$y^2 = \left(x + \frac{1}{64}\right)^2 \left[-4 \left(x + \frac{1}{64}\right)^2 + f(x)^2 \right] \quad (\text{E.28})$$

at the singularity. The double zero at $x_0 \simeq 1$ come from (as can be seen checked explicitly):

$$f(x) + 2 \left(x + \frac{1}{64}\right) = (x - x_0)^2. \quad (\text{E.29})$$

Near the singularity the curve looks like

$$y^2 = -4 \left(x + \frac{1}{64}\right)^4 + \left[\left(x + \frac{1}{64}\right) f(x) - \epsilon \right]^2 \quad (\text{E.30})$$

($u - u_0 \sim \epsilon$, $v - v_0 \sim \epsilon$.) Near $x = x_0 \simeq 1$, the double zero is split by the presence of terms linear in ϵ :

$$(x - x_0)^2 - 2\epsilon \left(x + \frac{1}{64}\right) f(x) \simeq 0 \quad (\text{E.31})$$

that is:

$$x \simeq x_0 \pm 2\epsilon^{1/2}. \quad (\text{E.32})$$

In conclusion, a small 2π circle around the singularity implies that the two branch points (double zero splitted) simply exchange between themselves. Again these movements of the branch points have been confirmed by a numerical analysis (1.10 in (Fig. 12)).

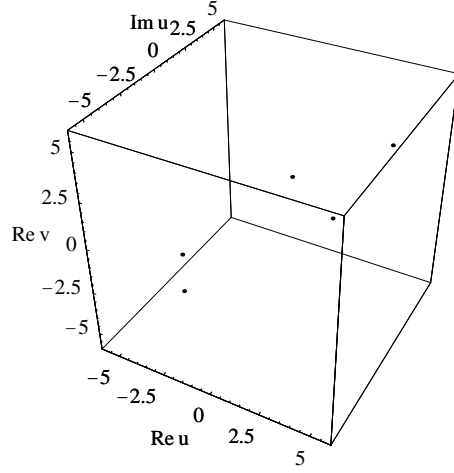


Figure 1: Five vacua of the $SU(3)$, $N = 1$ theory with $n_f = 1$ flavors, plotted as the projection $(\text{Re } u, \text{Im } u, \text{Re } v)$ of the QMS. ($\Lambda_1 = 2$, $m = 1/64$, $u \equiv \frac{1}{2}\text{Tr}\langle\Phi^2\rangle$, $v \equiv \frac{1}{3}\text{Tr}\langle\Phi^3\rangle$).

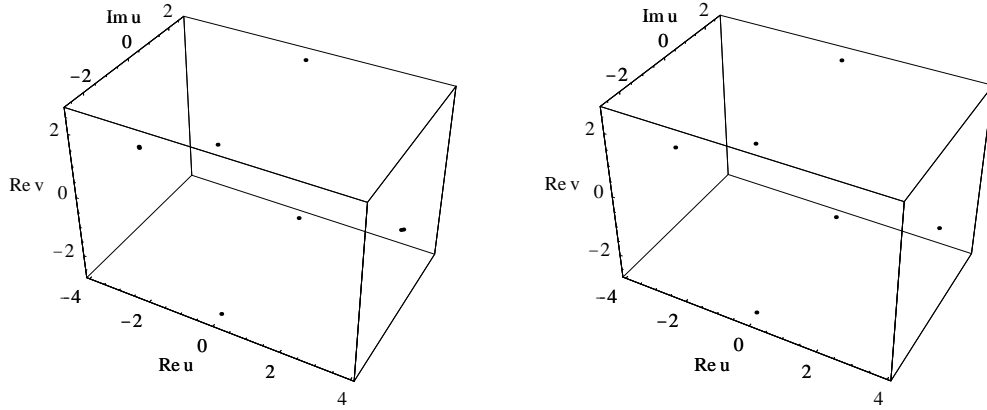


Figure 2: In the left figure are the eight vacua of the $SU(3)$ theory with $n_f = 2$, plotted as the projection $(\text{Re } u, \text{Im } u, \text{Re } v)$ of the QMS. ($\Lambda_2 = 2$, $m_1 = 1/64$, $m_2 = i/64$). The same in the right figure with equal masses $m_1 = m_2 = 1/64$.

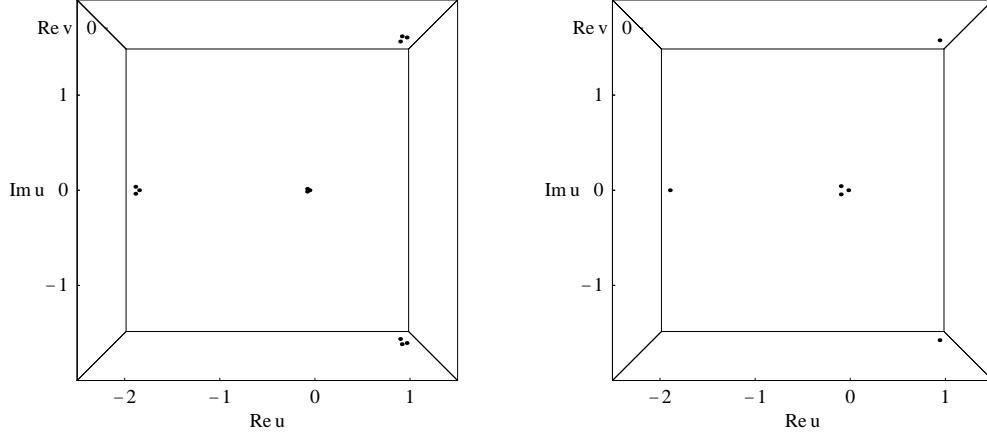


Figure 3: Twelve vacua of the $SU(3)$ theory with $n_f = 3$ in the projection $(\text{Re } u, \text{Im } u, \text{Re } v)$. $\Lambda_3 = 2$, $m_1 = 1/64$, $m_2 = i/64$, $m_3 = -i/64$. The same projection in the right with equal masses: $m_i = 1/64$.

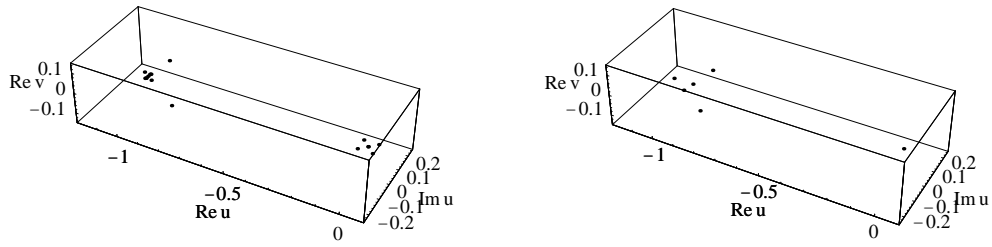


Figure 4: The seventeen vacua of the $SU(3)$ theory with $n_f = 4$ in the $(\text{Re } u, \text{Im } u, \text{Re } v)$ projection. $\Lambda_4 = 2$, $m_1 = 1/64$, $m_2 = -1/64$, $m_3 = i/64$, $m_4 = -i/64$. On the right, the same plot in the equal masses case with $m_i = 1/64$.

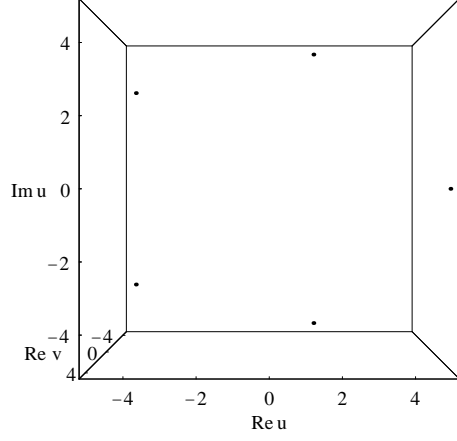


Figure 5: Five vacua of the $USp(4)$ with $n_f = 1$ flavors, plotted in $(\text{Re}u, \text{Im}u, \text{Re}v)$ projection. $\Lambda_1 = 2^{1/5}$, $m = 1/64$, while $u = \phi_1^2 + \phi_2^2$ and $v = \phi_1^2 \phi_2^2$.

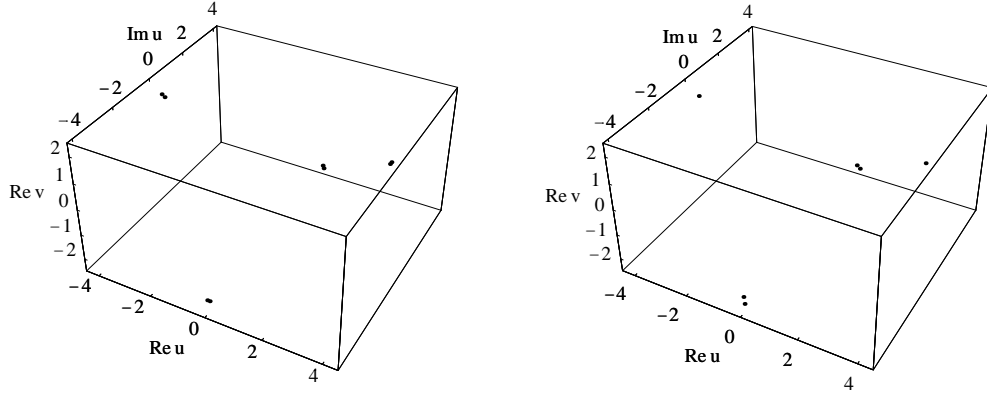


Figure 6: In the left figure are the eight singularities of the $USp(4)$ theory with $n_f = 2$ are plotted in the $(\text{Re}u, \text{Im}u, \text{Re}v)$ projection. $\Lambda_2 = 2^{1/4}$, $m_1 = 1/64$, $m_2 = i/64$. In the right, the same plot with equal masses ($m = 1/64$).

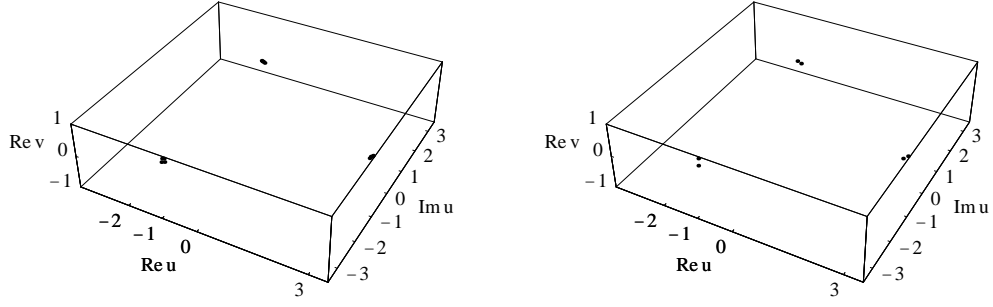


Figure 7: The twelve singularities of the $USp(4)$ theory with $n_f = 3$, plotted in the $(\text{Re } u, \text{Im } u, \text{Re } v)$ projection of the QMS. $\Lambda_3 = 2^{1/3}$, $m_1 = 1/64$, $m_2 = i/64$, $m_3 = i/256$. On the right, the same plot with equal masses ($m = 1/64$).

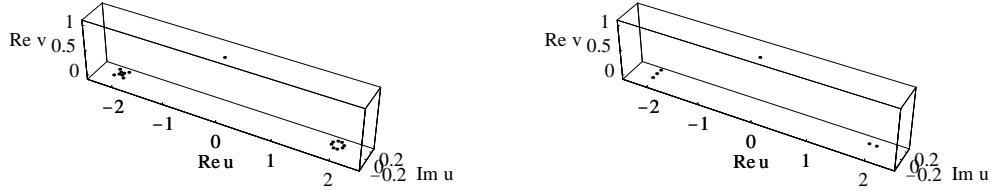


Figure 8: The seventeen vacua of the $USp(4)$ with $n_f = 4$, plotted in the $(\text{Re } u, \text{Im } u, \text{Re } v)$ projection. $\Lambda_4 = \sqrt{2}$, $m_1 = 1/64$, $m_2 = i/64$, $m_3 = 1/32$, $m_4 = i/32$. On the right, the same plot in the case of equal masses, $m_i = 1/64$, $\forall i$.

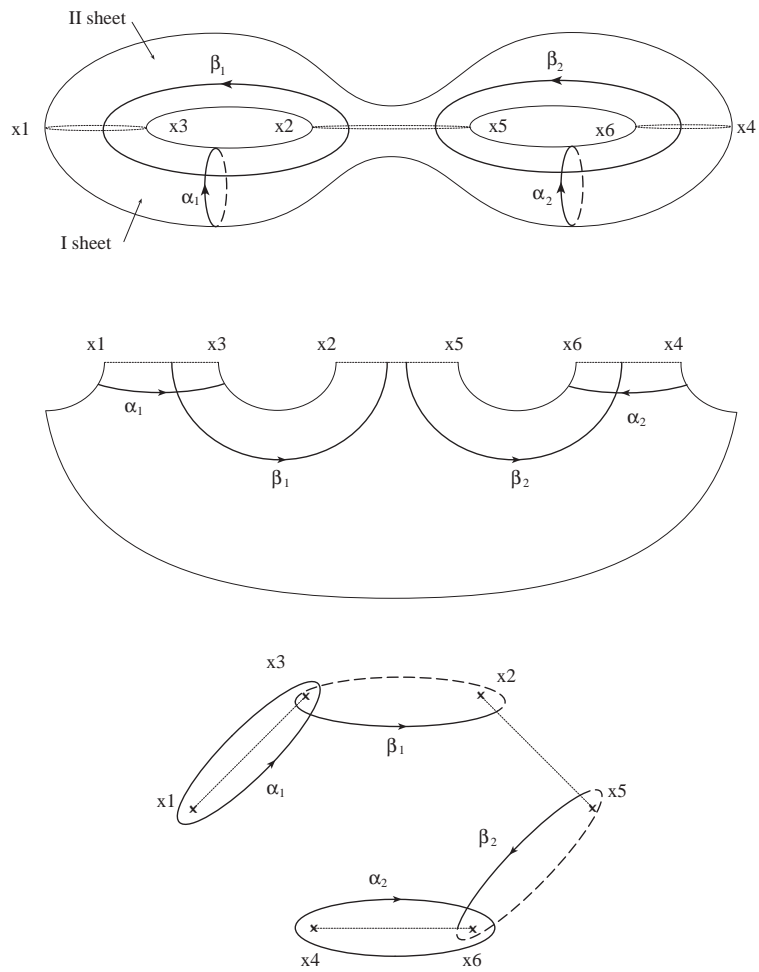
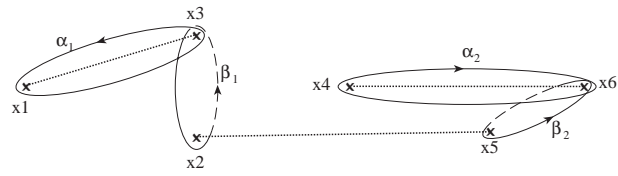


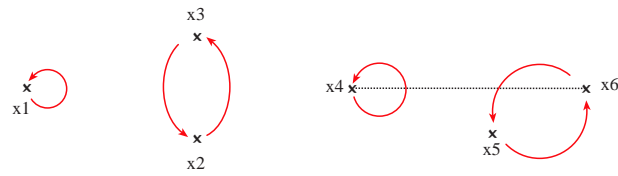
Fig. 1.0

Figure 9:

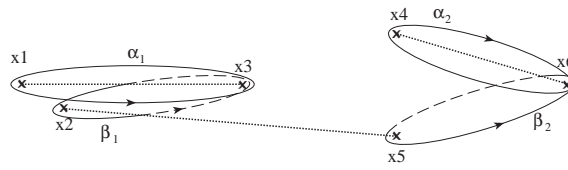
1.1



1.2



1.3



1.4

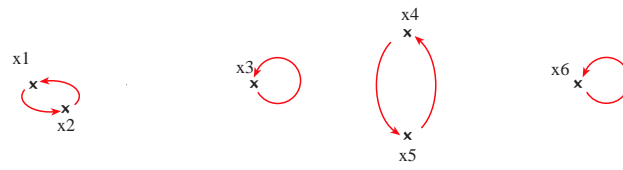


Figure 10:

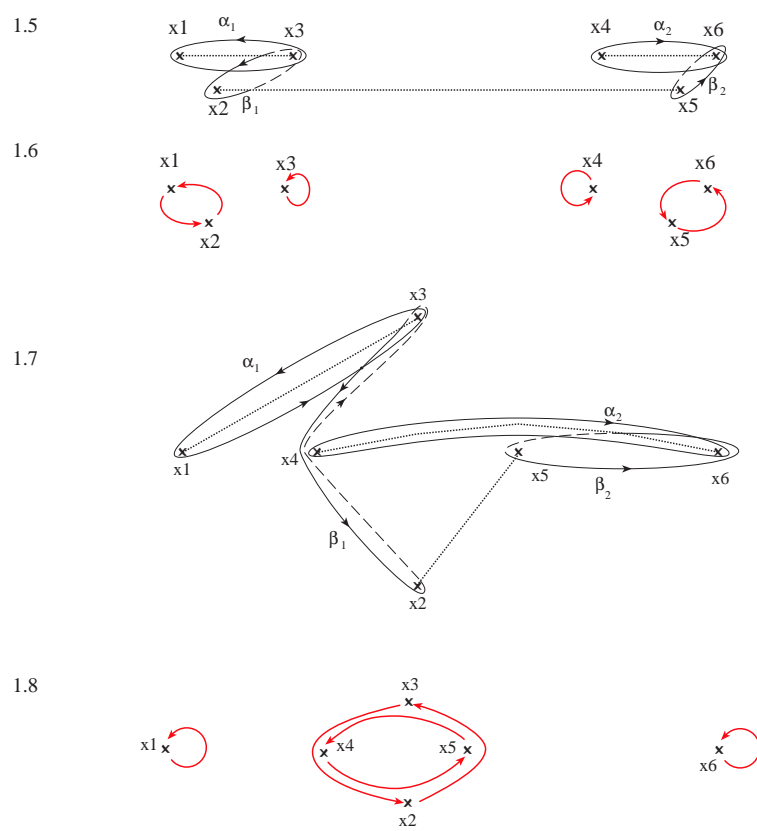


Figure 11:

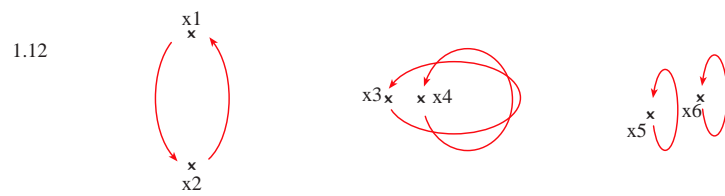
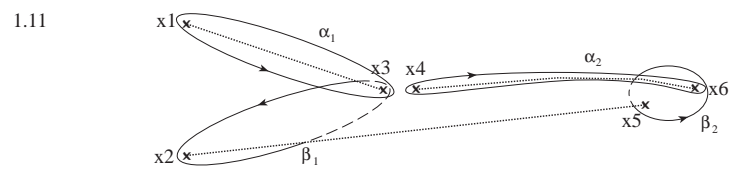
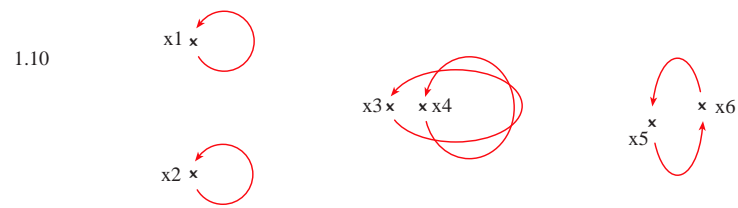
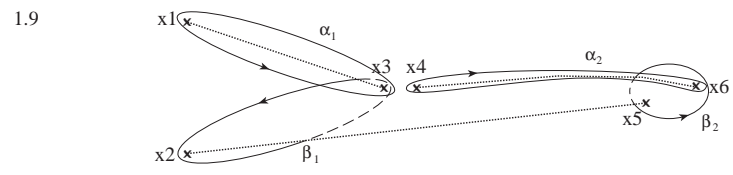


Figure 12: